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GRAPHS WITH SMALL BANDWIDTH AND CUTWIDTH

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We give counter-examples to the following conjecture which arose in the study of small bandwidth graphs.

"For a graph G, suppose that $|V(G')| \leqslant 1 + c_1 \cdot \text{diameter } (G')$ for any connected subgraph G' of G, and that G does not contain any refinement of the complete binary tree of c_2 levels. Is it true that the bandwidth of G can be bounded above by a constant c depending only on c_1 and c_2?"

On the other hand, we show that if the maximum degree of G is bounded and G does not contain any refinement of a complete binary tree of a specified size, then the cutwidth and the topological bandwidth of G are also bounded.

1. Introduction

For a graph G with vertex set $V(G)$ and edge set $E(G)$, a *numbering* of G is a one-to-one mapping π from $V(G)$ to the integers. The *bandwidth* of a numbering π is

$$\max\{|\pi(u) - \pi(v)| : \{u, v\} \in E(G)\}.$$

The *bandwidth* $b(G)$ of G is the minimum bandwidth of all numberings. The *cutwidth* of a numbering π is

$$\max_i |\{\{u, v\} \in E(G) : \pi(u) \leqslant i < \pi(v)\}|.$$

The *cutwidth* $c(G)$ of G is the minimum cutwidth of all numberings.

The bandwidth problem and the cutwidth problem are associated with many optimization problems in circuit layout. In a circuit design or a network system, the maximum length of the wire is often proportional to the delay for transmitting messages, and so bandwidth is a graph-invariant of importance in circuit design. On the other hand, the cutwidth problem is of particular interest in designing microchip circuits and is often associated with the area for the layout (see [7]).

One of the interesting problems about bandwidth is to understand what substructures force up the bandwidth of a graph. There are two known factors which may make bandwidth large. The first is the density lower bound (see [1, 2]):

$$b(G) \geqslant \frac{|V(G)| - 1}{D(G)}$$

where $D(G)$ is the diameter of G, that is, the maximum distance among all pairs of vertices in G. A somewhat stronger lower bound, the so-called "local density" bound, can also be easily obtained:

$$b(G) \geq \max_{G' \subseteq G} \frac{|V(G')| - 1}{D(G')},$$

where G' ranges over all connected subgraphs of G with ≥ 2 vertices. One natural problem arises: "If local density is small, is it true that the bandwidth is small?" This question was answered in the negative by Chvátalová [4] by examining refinements of the complete binary tree B_k of k levels. A graph G' is said to be a *refinement* of G if G' can be formed by replacing some edges in G by paths. For each integer k, every refinement of B_{2k} has bandwidth $\geq k$, and there is a refinement of B_{2k} with local density at most 3.

Now if a graph contains a refinement of B_{2k}, its bandwidth is at least k. Again, containing a large complete binary tree is sufficient but not necessary for the graph to have large bandwidth (as we see from the star $K_{1,n}$.) That suggests the following question. Suppose that the local density of a graph G is no more than c_1, and that G does not contain any refinement of B_{c_2}. Is it true that the bandwidth of G is bounded above by a constant depending only on c_1 and c_2?

We mention that Chvátalová and Opatriný [5] proved a somewhat similar result for infinite trees. They showed that if an infinite countable tree T satisfies that (i) the maximum degree is at most c_1, (ii) the number of edge-disjoint semi-infinite paths is at most c_2, and (iii) T does not contain a refinement of B_{c_3} as a subgraph, then some refinement of T has finite bandwidth bounded above by a function depending only on c_1, c_2 and c_3.

In this paper, we will prove two results. One of them answers the above question in the negative and identifies a third structure which drives up the bandwidth. The other result answers positively an analogous question for cutwidth (or "topological bandwidth", defined below).

Theorem 1. *For each integer k, there exists a tree with the following properties:*
(i) *its local density is at most* 9
(ii) *it does not contain any refinement of* B_4
(iii) *its bandwidth is at least k.*

Theorem 2. *Suppose that G has maximum degree* c_1, *and does not contain any refinement of* B_{c_2}. *Then the cutwidth of G is bounded above by a constant depending only on* c_1 *and* c_2.

The *topological bandwidth* $b^*(G)$ of a graph G is the minimum bandwidth $b(G')$ over all refinements G' of G. The topological bandwidth problem can be viewed as the optimization problems of circuit layout when vertices of degree two (interpreted as "drivers" or "repeaters") can be inserted to help minimize the length of the edges. Cutwidth and topological bandwidth are known to be closely related, and it has been shown [3, 6] that $b^*(G) \leqslant c(G)$ for any graph G. In particular, for trees [3]

$$b^*(T) \leqslant c(T) \leqslant b^*(T) + \log_2 b^*(T) + 2.$$

But it is not hard to see that for some graphs, such as $G = K_n$, the cutwidth $c(G)$ can be much larger than $b^*(G)$. Nevertheless, Theorem 2 implies the following relation between $c(G)$ and $b^*(G)$.

Theorem 3. *There is a function f such that for any graph G, $c(G) \leqslant f(b^*(G))$.*

(One interpretation of Theorem 3 is that if the topological bandwidth is bounded above by a constant c_1, then the cutwidth is bounded above by another constant c_2 which depends only on c_1.)

The paper is organized as follows. In Section 2, we will construct some special trees, the so-called Cantor combs, which imply Theorem 1. In Section 3 we will give the proof of Theorems 2 and 3.

2. Cantor combs

In this section, we will show that the two conditions, local density $\leqslant c_1$ and containing no binary tree of c_2 levels, do not imply small bandwidth.

A *comb* is a tree T with two special vertices, called its *roots*, such that every vertex of T with degree $\geqslant 3$ has degree 3 and lies on the path of T between the roots. For $k \geqslant 1$, we define the *Cantor comb* C_k as follows. C_1 is the 2-vertex tree, where both vertices are roots. Inductively, having defined C_{k-1}, we define C_k as follows. Take two disjoint copies T_1, T_2 of C_{k-1} with roots s_1, t_1 and s_2, t_2. Let P and Q be paths with $4\,|V(C_{k-1})|$ and $6(k-1)\,|V(C_{k-1})|$ edges respectively, such that P, Q, T_1 and T_2 are mutually vertex-disjoint except that p has ends t_1 and t_2, and one end of Q is the middle vertex of P. We define C_k to be $T_1 \cup T_2 \cup P \cup Q$, with roots s_1, s_2. This completes the inductive definition of C_k. We observe that there is an automorphism of C_k exchanging the roots. Let $|V(C_k)| = N_k$ $(k \geqslant 1)$. We shall show that C_k satisfies Theorem 1, by means of the following assertions.

(2.1) *For $k \geqslant 1$, the bandwidth of C_k is at least k.*

Proof. If possible, choose $k \geqslant 1$ minimum such that C_k has bandwidth $<k$. Then $k \geqslant 2$,; let T_1, T_2, P, Q etc. be as in the definition of C_k. Let π be a numbering of C_k with bandwidth $\leqslant k-1$. Since T_1 and T_2 both have bandwidth $\geqslant k-1$ from the minimality of k, it follows that for $i = 1, 2$ there is an edge $e_i = \{u_i, v_i\}$ of T_i such that $\pi(v_i) - \pi(u_i) = k-1$ and every integer between $\pi(u_i)$ and $\pi(v_i)$ equals $\pi(w)$ for some $w \in V(T_i)$. Choose $w_i \in \{u_i, v_i\}$ $(i = 1, 2)$ so that the path R of C_k

between w_1 and w_2 uses neither e_1 nor e_2. We may assume that $\pi(w_1) < \pi(w_2)$. Since $|E(P)| = 4N_{k-1}$ it follows that $|E(R)| < 6N_{k-1}$ and so $\pi(w_2) - \pi(w_1) < 6(k-1)N_{k-1}$. Since $|V(Q)| \geq 6(k-1)N_{k-1}$, some vertex $w \in V(Q)$ does not satisfy $\pi(w_1) < \pi(w) < \pi(w_2)$, and we may assume that $\pi(w) < \pi(w_1)$. Let S be the path of C_k between w and w_2. Since $\pi(w) < \pi(w_1) < \pi(w_2)$ and $w_1 \notin V(S)$, there are consecutive vertices u, v of S with $\pi(u) < \pi(w_1) < \pi(v)$. Since $\pi(v) - \pi(u) \leq k-1$ (because u, v are adjacent) it follows that $\pi(v) - \pi(w_1) < k-1$ and $\pi(w_1) - \pi(u) < k-1$; but then one of $\pi(u)$, $\pi(v)$ lies between $\pi(u_1)$ and $\pi(v_1)$, a contradiction. This completes the proof. □

For $k \geq 1$, we define L_k to be the number of edges in the path of C_k between its roots. Let v be a root; for $r \geq 0$ we define $X_k(r)$ to be the number of vertices of C_k different from v and within distance r of v. (From the symmetry of C_k, this does not depend on the choice of v.)

(2.2) *For $k \geq 1$, $X_k(r) \leq 3r$ if $r \leq L_k$, and $X_k(r) \leq 2r$ if $r > L_k$.*

Proof. We proceed by induction on k. The result holds for $k = 1$, and we assume $k > 1$. Let T_1, T_2, P, Q etc. be as in the definition of C_k.

(1) *If $r \leq L_{k-1}$ then $X_k(r) \leq 3r$.*

For every vertex of C_k within distance r of s_1 belongs to T_1, and the result follows from the inductive hypothesis.

(2) If $L_{k-1} < r < \frac{1}{2}L_k$ then $X_k(r) \leq 3r$.

For the number of vertices of T_1 within distance r of s_1 is at most $2r$, from our inductive hypothesis; and there are at most r further vertices of C_k within distance r of s_1, all from P.

(3) If $\frac{1}{2}L_k \leq r \leq L_k - L_{k-1}$ then $X_k(r) \leq 3r$.

For within distance r of s_1 there are at most N_{k-1} vertices of T_1, at most r further vertices of P, at most r further vertices of Q, and none from T_2. Thus

$$X_k(r) \leq N_{k-1} + 2r \leq 3r$$

since $r \geq \frac{1}{2}L_k \geq N_{k-1}$.

(4) If $r > L_k - L_{k-1}$ then $X_k(r) \leq 2r$.

For $P \cup T_1 \cup T_2$ has $\leq 6N_{k-1}$ vertices, and there are $r - \frac{1}{2}L_k \leq r - 2N_{k-1}$ further vertices of Q within distance r of s_1. Thus

$$X_k(r) \leq 6N_{k-1} + r - 2N_{k-1} \leq 2r$$

since $r \geq L_k - L_{k-1} \geq |E(P)| = 4N_{k-1}$.

This completes the proof of (2.2). □

(2.3) *Let $k \geq 1$ and $r \geq 0$ be integers and let $v \in V(C_k)$. There are at most $9r + 1$ vertices of C_k within distance r of v.*

Proof. We proceed by induction on k. We may assume that $k \geq 2$. Let T_1, T_2, P, Q, etc. be as before. Now there are three paths of $C(v)$, each starting at v, which include $P \cup Q$ in their union, and at most r vertices different from v of each of these paths is within distance r of v. Thus at most $3r$ vertices of $P \cup Q$ are different from v and are within distance r of v. If $v \in V(P \cup Q)$ then, by (2.2), for $i = 1, 2$ at most $3r$ vertices of T_i are different from v and are within distance r of v, as required. Thus we may assume that $v \in V(T_1) - V(P)$. Let the number of edges in the path of T_1 from v to t_1 be L. If $L \geq r$ the result follows from our inductive hypothesis applied to T_1. Thus we assume that $L < r$. Every vertex of T_1 within distance r of v is within distance $r + L$ of t_1; and every vertex of T_2 within distance r of v is within distance $r - L$ of t_2. Thus by (2.2), there are at most $3(r + L) + 3(r - L)$ vertices of $T_1 \cup T_2$ different from t_1, t_2, v which are within distance r of v. Hence in total there are at most $9r + 1$ vertices of C_k within distance r of v as required. □

From assertions (2.1) and (2.3) we deduce the following result, which implies Theorem 1.

Theorem 1′. *For $k \geq 1$, the comb C_k has local density ≤ 9, it does not contain any refinement of B_4, and its bandwidth is at least k.*

Proof. Let G' be a connected subgraph of C_k with $|V(G')| \geq 2$. Choose $v \in V(G')$. By (2.3), $|V(G')| \leq 9D(G') + 1$ and so

$$\frac{|V(G')| - 1}{D(G')} \leq 9.$$

Thus C_k has local density ≤ 9. Moreover, it contains no refinement of B_4 since it is a comb, and its bandwidth is at least k by (2.1). □

3. Bounded cutwidth or topological bandwidth

Before we proceed to prove that having bounded degree and containing no refinement of some bounded complete binary tree imply bounded cutwidth and hence bounded topological bandwidth, we will first discuss the "path-width" of a graph, which was introduced in [8] for studying graph minors. The *path-width* of a graph G is the minimum $k \geq 0$ such that its vertex set $V(G)$ is a union of subsets $V_1, V_2, \ldots, V_t$ with the following properties;

(i) $|V_i| \leq k + 1$ for $1 \leq i \leq t$.

(ii) $V_i \cap V_j \subseteq V_m$ for $1 \leq i \leq m \leq j \leq t$

(iii) for each edge $\{u, v\}$, there exists some V_i containing both u and v.

Path-width and bandwidth can differ significantly; for example, a star $K_{1,n}$ has path-width ≤ 1 and bandwidth $\geq \frac{1}{2}n$.

In [8] it was shown that if a graph contains no refinement of B_{c_1} then its path-width is at most c_2, where c_2 depends only on c_1. This will be used to prove Theorem 2.

Proof of Theorem 2. Since G contains no refinement of B_{c_2}, its path-width is at most c_3 where c_3 depends only on c_2. Let $V_1, V_2, \ldots, V_t$ denote subsets of G with $|V_i| \leq c_3 + 1$ $(1 \leq i \leq t)$, as in the definition of path-width. For each vertex v, we define $a(v)$ and $b(v)$ to be respectively the least and largest numbers i such that v is in V_i. Choose a numbering π from $V(G)$ to integers $\{1, 2, \ldots, |V(G)|\}$ such that $\pi(u) \leq \pi(v)$ if and only if $a(u) \leq a(v)$. (Ties in $a(v)$ are broken in any arbitrary way.) We shall show that π (and hence G) has cutwidth $\leq c_1(c_3 + 1)$. Let i be any number between 1 and $n = |V(G)|$. Choose $x \in V(G)$ with $\pi(x) = i$. We claim that $u \in V_{a(x)}$ for every edge $\{u, v\}$ with $\pi(u) \leq i < \pi(v)$. For $a(u) \leq a(x)$ since $\pi(u) \leq \pi(x)$, and $a(x) \leq a(v)$ since $\pi(x) \leq \pi(v)$. Moreover, $a(v) \leq b(u)$ since $\{u, v\}$ is an edge. Hence $a(u) \leq a(x) \leq b(u)$ and consequently $u \in V_{a(x)}$, as claimed. But there are at most $c_3 + 1$ vertices in $V_{a(x)}$ each of which is adjacent to at most c_1 vertices. So there are at most $c_1(c_3 + 1)$ edges "crossing" i, that is,

$$|\{\{u, v\} \in E(G) : \pi(u) \leq i < \pi(v)\}| \leq c_1(c_3 + 1),$$

for every i. This completes the proof of Theorem 2. □

Armed with Theorem 2 it is easy to deduce Theorem 3.

Proof of Theorem 3. Let G be a graph with $b^*(G) = k$. Then G contains no refinement of B_{2k+2} (since every such refinement has bandwidth $\geq k + 1$) and G has maximum degree $\leq 2k + 1$. From Theorem 2, $c(G)$ is at most some $f(k)$, as required. □

We conclude by proposing the following problem.

For a graph G, suppose all subtrees of G have bandwidth $\leq c$. Is it true that the bandwidth of G is no more than c' where c' is a function of c?

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