

# Herbert S. Wilf (1931-2012)<sup>1 2</sup>

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Herbert S. Wilf, Thomas A. Scott Emeritus Professor of Mathematics at the University of Pennsylvania, died on January 7, 2012 in Wynnewood, PA, of amyotrophic lateral sclerosis (ALS). He taught at Penn for 46 years, retiring in 2008. He was widely recognized both for innovative research and exemplary teaching: In addition to receiving other awards, he is the only mathematician to have received both the Steele Prize for Seminal Contributions to Research (from the A.M.S., 1998) and the Deborah and Franklin Tepper Haimo Award for Distinguished Teaching (from the M.A.A., 1996). During his long tenure at Penn he advised 26 Ph.D. students and won additional awards, including the Christian and Mary Lindback Award for excellence in undergraduate teaching. Other professional honors and awards included a Guggenheim Fellowship in 1973-74, and the Euler Medal, awarded in 2002 by the Institute for Combinatorics and its Applications.

Herbert Wilf's mathematical career can be divided into three main phases. First was numerical analysis, in which he did his Ph.D. dissertation (under Herbert Robbins at Columbia University in 1958) and wrote his first papers. Next was complex analysis and the theory of inequalities, in particular, Hilbert's inequalities restricted to  $n$  variables. He wrote a cluster of papers on this topic, some with de Bruijn [1] and some with Harold Widom [2]. Wilf's principal research focus during the latter part of his career was combinatorics. In 1965, Gian-Carlo Rota came to the University of Pennsylvania to give a colloquium talk on his then-recent work on Möbius functions and their role in combinatorics. Wilf recalled, "That talk was so brilliant and so beautiful that it lifted me right out of my chair and made me a combinatorialist on the spot." The next year, Wilf took a sabbatical leave in London, where he spent his time in the library at Imperial College exploring combinatorics by reading "every issue of *Math Reviews* starting from the creation of the world." By the end of the year, he had proved a result [4] which he described as his "debut": The chromatic number of a graph  $G$  is at most 1 plus the largest eigenvalue of its adjacency matrix, with equality if and only if  $G$  is a complete graph or an odd circuit.

In fact an earlier paper [3], written with N. J. Fine, shows signs of Wilf's developing interest in combinatorics. It contains the following result: If two infinite periodic sequences  $\{a_n\}$  and  $\{b_n\}$  have periods  $p$  and  $q$  respectively, and  $a_n = b_n$  for  $p + q - \gcd(p, q)$  consecutive integers  $n$ , then  $a_n = b_n$  for all  $n$ . Furthermore,  $p + q - \gcd(p, q)$  is the smallest possible value making the statement true. This result and its continuous generalizations became collectively known as the "Fine-Wilf Theorem," and [3] has been widely cited in both the math and computer science literature.

From then on, Wilf had a profound influence in numerous areas of combinatorics, including graph theory, discrete algorithms, enumeration, asymptotics, and generating functions. Wilf's pioneering work [5] with Doron Zeilberger on computerized discovery and proof of hypergeometric identities resulted in the book  $A = B$  (with Marko Petkovšek), as well as the A.M.S. Steele Prize citation in 1998. In addition, Wilf made notable contributions to the study of permutation

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<sup>1</sup>The text of this article draws from several sources, including a tribute written by Chung and Hutchinson for Wilf's 65th birthday celebration in 1996 (see [9]) and also essential input from Ruth Wilf, who also took most of this article's photos. Some of the material in quotes is drawn from Wilf's personal notes.

<sup>2</sup>This is a copy of a memorial to appear in the A.M.S. Notices with this version containing all references used by the contributing authors.

patterns, for example, the Stanley-Wilf conjecture (see [6]) which stimulated much interest until it was settled in 2004 by Marcus and Tardos [21]. Much of his later work was on generating functions and enumerative problems, and he was especially known for his work on integer partitions. For example, his typically succinct and gem-like paper with Neil Calkin [7] contains this result: The positive rational numbers are uniquely enumerated by the sequence  $\{b(n)/b(n-1)\}$ , where  $b(n)$  is equal to the number of ways of writing  $n$  as a sum of powers of 2, each repeated at most twice. To prove this theorem, the authors invent a structure which subsequently became famous in its own right as the “Calkin-Wilf tree.”

While still an undergraduate at M.I.T., Wilf began taking summer jobs at Nuclear Development Associates, which did consulting work for utility companies trying to build electric-power-generating nuclear reactors. He described the young staff as “incredibly brilliant and idealistic as they faced the new frontier.” Wilf recalled being greatly influenced by Gerald Goertzel, a physicist then at N.Y.U., and J. Ernest Wilkins, a pioneering black mathematician and physicist who later enjoyed a distinguished academic career at Howard and Clark Atlanta Universities. After completing his Ph.D. coursework at Columbia, Wilf worked full-time for NDA and later for Fairchild Engine Division on Long Island, where he headed the computing section, with a small staff of one (himself). Wilf’s work at Fairchild included calculating eigenvalues of a proposed jet engine design on the IBM Card Programmed Calculator (CPC).

Long before most researchers and educators, Wilf recognized the important connections between computers and mathematics. In the summer of 1954 he worked for IBM in Manhattan, where the first mainframe IBM 701 had recently been unveiled. He and his M.I.T. roommate Tony Ralston worked in the display window of IBM’s headquarters on Madison Avenue, an experience he described as “heady stuff.” They spent the first month learning how to program, since neither had ever seen a computer before. Eventually, they wrote a program implementing a multistage linear regression analysis, following a method developed by Mike Efroymsen, then of Esso (now Exxon) Corp. The program was used by Esso to control the quality of refined gasoline. Together, Wilf and Ralston edited the first volume of *Mathematical Methods for Digital Computers*, followed by two more volumes a few years later.

After receiving his Ph.D. from Columbia in 1958, Wilf taught for three years at the University of Illinois. In 1962, after visiting Penn during a yearlong celebration of Hans Rademacher’s 70th birthday, he received an offer to stay on permanently, which he accepted.

Wilf was a superb teacher. His lectures were full of delight, always finding the easy and graceful way to reveal the inner workings of mathematics. Wilf’s philosophy in teaching was reflected in his article “*On buckets and fires*” [8], named after the quote “A mind is a fire to be kindled, not a bucket to be filled”. Throughout his years at Penn, Wilf lit that fire in numerous undergraduates, graduate students, colleagues, friends and other researchers. In particular, his Ph.D. students have directly benefitted not only by his teaching but also from his example of how to be a mathematician.

The computer continued to play a key role in both Wilf’s research and teaching. He and Albert Nijenhuis masterfully integrated FORTRAN with combinatorics as seen in their book *Combinatorial Algorithms*. Wilf’s interest in computers continued to grow and flourish in his celebrated work with Petkovšek and Zeilberger, which employed computers in a novel way and probed the boundaries of what had previously been recognized as mathematical proof.

Wilf co-founded two major journals, *Journal of Algorithms* in 1980 with Donald Knuth, and the *Electronic Journal of Combinatorics* in 1994 with Neil Calkin, for which he served as an Editor-

in-Chief. During his tenure as the editor for the *American Mathematical Monthly* (1987-1991), he contributed a series of illuminating expository articles, in addition to fulfilling his editorial duties. In his long and prolific research career, Wilf published seven books and more than 160 research articles, collaborating in the process with more than 60 mathematicians. He was an early advocate of open access to books and journals, and worked with his publishers to make most of his books available for free downloading from his Penn website <http://www.math.upenn.edu/~wilf/>. Many of his books remain accessible there, including *Mathematical Methods for the Physical Sciences*, *Combinatorial Algorithms*, *Algorithms and Complexity*, *generatingfunctionology*, and  $A = B$ .

Wilf remained deeply engaged with mathematics after retiring from Penn, teaching a course regularly, publishing 10 papers in the last five years of his life, and collaborating intensively with research colleagues by e-mail even when illness limited his mobility. Throughout his career he was a highly regarded lecturer, and traveled extensively giving invited talks at conferences and colloquia throughout the world. In May 2011 he attended an 80th birthday conference in his honor at Wilfred Laurier University in Waterloo, Ontario. At that conference he gave a talk on two recent papers with Penn Biology colleague Warren Ewens on combinatorial and statistical problems in mathematical biology [10, 11], published in the *Proceedings of the National Academy of Sciences*. An earlier conference had been held in 1996 at the University of Pennsylvania to celebrate his 65th birthday (see [9]).

The intensity and diversity of Wilf's interests extended far beyond mathematics. For about 20 years, until 2002, he was an avid amateur pilot, often flying himself to conferences and mathematical events in his single-engine Cessna. His travels as a pilot included many coast-to-coast flights in the U.S., and trips with his wife to Alaska, Baja California, Nova Scotia, Newfoundland, and the Bahamas, among other places. Hobbies to which he devoted himself at various times with similar passion included large-format photography and developing, astronomy and telescopes, bread making, bird watching, and boating.

Wilf is survived by his wife Ruth Tumen Wilf, whom he married one month after receiving his B.S. degree from M.I.T. in 1952. Within four years they had two children, Susan and David, followed by Peter seven years later. While juggling family responsibilities and finishing his own Ph.D. at Columbia, he also provided moral and other support to Ruth as she completed her Ph.D. in Zoology at Columbia. She subsequently shifted gears to become a certified nurse-midwife, and is now only partially retired after enjoying a long and active professional career. Wilf was devoted to his family and took great interest in his children and six grandchildren.

Mathematical friends, colleagues, and students inevitably became the Wilfs' friends, and were beneficiaries of their gracious hospitality, either at home in Narberth, Pennsylvania, or at their shore house in New Jersey. On these occasions, the Wilfs often delighted in taking guests on boating expeditions on Barnegat Bay to spot birds and other wildlife, and at night to stargaze and watch meteor showers.

Herb Wilf's playful humor, warmth, and generosity were essential ingredients of his personal and mathematical interactions of all kinds. These qualities resonate throughout the remembrances of his mathematical friends and collaborators which accompany this article.



*Photo #1: Herb Wilf, Thanksgiving 2009.*



*Photo #2: Herbert Wilf, high school graduation photo, age 16.*



*Photo #3: Herb and Ruth Wilf with his beloved Cessna, early 1990's.*

## *Donald E. Knuth*

Herb Wilf was a giant in many ways. Not only was he the tallest mathematician I ever met, he also stood out as one of the "most significant figures" in my own life. As I write these reminiscences, three months after his untimely death, I'm astonished at how many times each day I still am reminded of things that I learned from him. He taught me about mathematics, he taught me about pedagogy, he taught me about leadership, he taught me about living life to the fullest, and he introduced me to some of my favorite jokes. Alas, I'm forced now to realize that the number of things I'll learn from him is bounded above. But my good fortune has been that the lower bound is already quite high.

I actually met Herb rather late in life, although of course I'd heard his name rather often before we ever connected. In the spring of 1974, he published an abstract announcing a new way to generate uniformly random partitions of a given integer, and I wrote to ask for a preprint. By the fall of that year he had sent me a complete draft manuscript of his book with Albert Nijenhuis, *Combinatorial Algorithms*, and our decades of correspondence began in earnest. By the 1990s we were writing letters back and forth about once per month, mostly trading conjectures or solutions to problems of common interest.

At the beginning of 1979 he presented me a sweet offer: He foresaw the need for a new journal, to be entitled *Journal of Algorithms* (a perfect title!), and he asked me to serve as founding co-editor although he promised to do 99% of the work. Naturally I agreed. Typical of my own involvement was the following quote from a letter that I sent him in April: "In any case I wish to leave the final decisions about these matters to you, and I will sign what you tell me to. (If the typography of the journal turns out to be unacceptable, I'll scream though.)" His vision was indeed prophetic, and *JoA* immediately began to thrive.

One of my fond memories from the 80s is the time that he sent me a bouquet of roses. I had submitted a paper under a pseudonym — a female name, in fact — in hopes of getting referee reports that were based entirely on the content of my work rather than on my inflated reputation. Herb agreed to this experiment (and later did a similar thing with a paper of his own that he submitted to the *American Mathematical Monthly*). When he wrote to my alter ego, acknowledging receipt of the manuscript, he added "By the way, if you're free next Saturday night, I have a couple of tickets to ...". And the roses arrived after the paper was accepted.

He also contributed a priceless guest lecture to a course called Mathematical Writing that I taught at Stanford in 1987. Among other tidbits, he demonstrated how to write up a typical theorem in two dramatically different styles: One, for *Acta Hypermathica*, was called "Enumeration of orbits of mappings under action of  $C_n$ , the cyclic group"; the other, for *MathWorld*, was called "Counting necklaces." (See [22].)

We enjoyed submitting a paper together [12] to Crelle's venerable *Journal für die reine und angewandte Mathematik*, extending a result that E. E. Kummer had contributed to the same publication some 137 years earlier.

Herb's continuing foresight led him to launch one of the very first eJournals, the *Electronic Journal of Combinatorics*, in 1994. Although 'way ahead of its time, it was another magnificently successful venture.

Let me close with one other indelible memory. On the occasion of my 64th birthday — that's the BIG ONE for a computer scientist! — Herb was the keynote speaker, and he even presented me with an original theorem (See <http://www.math.upenn.edu/~wilf/website/tributes.html>).

Thus it's no surprise that, when I lectured to Stanford's theory students on January 19, 2012, I dedicated my talk to Herb's memory, and wore a yellow T-shirt that exhibits the Calkin–Wilf sequence [7].



Photo #4: Don Knuth, Herb Wilf, at Knuth's 64th birthday conference, 2002. (Photo by Jill Knuth)

## *Aviezri S. Fraenkel*

Herbert Saul Wilf was a great alchemist. Anything he touched turned into pure gold, be it spreading the word, applications of math, or pure math. Here are a few examples, out of many. *Spreading the word.* Herb recognized early the importance of the budding algorithmics era, and swiftly founded the J. of Algorithms, together with Don Knuth; he noticed the emerging internet and promptly pioneered, jointly with Neil Calkin, the Electronic Journal of Combinatorics, which became highly influential and boasts nowadays over a thousand submissions annually; he was the editor-in-chief of the *American Mathematical Monthly*, and on the advisory board of *Integers*, and the *Electronic Journal of Combinatorial Number Theory*.

*Applications of math.* We note Herb's two recent ventures into the life sciences, both of which lean heavily on math. For both he teamed up with biologist Warren Ewens. First there is the contribution to epidemiology [10], which suggests that a certain outbreak of childhood leukemia in a small community is not uncommon, with high probability; whereas the outbreak of acute lymphocytic leukemia in another small community cannot be explained by chance, with high probability. Secondly, there is the profound insight into the theory of genetic evolution [11], advancing bold new ideas of replacing in series by in parallel processes, thus removing the obstacle that evolutionary theories take too much time.

*Pure math and applications to pure math.* Herb saw Z and made it into the ground-breaking WZ, which got the authors an AMS Leroy P. Steele Prize for Seminal Contributions to Research; he had a look at the rationals and created the short, elegant and influential paper, joint with Neil Calkin [7] to count the rationals in a beautiful new way. This result even finds an unexpected application in game theory [23].

How come everything Herb touched became pure gold, having immediate yet lasting impact?

On the face of it, Herb's contributions to math applications aren't all that surprising. After all, Nobel laureate physicist Eugene Wigner wrote "The un-reasonable effectiveness of mathematics in the natural sciences" [24]. Math is being recognized more and more as the dominating ingredient

for successful research of an increasing number of disciplines. It's a *gate-opener* to the natural sciences.

However, not everybody has the keys to these gates. Most mathematicians have either a continuous soul or a discrete soul. Herb had an integrated global soul with a complete command of both. Being a master locksmith, he created the skeleton key for all those gates, while sharing a kind and feeling soul with his family, friends and mankind.

This is not the entire story. Herb had a passion and love for math, doing math constantly, much like Paul Erdős did, but without the eccentricity of the latter. His talent, elegance, precision, efficiency, inner strength and integrity, drawing on his global command of math were the rock foundations that guided him unfailingly into the truly promising directions in research as well as in spreading the word. These qualities enabled him to strike gold.

I visited him about two months before his untimely departure. His wife Ruth had made a major transformation, lovingly turning their home into a hospice- like ward to ease his condition as much as possible and enable him to work under the illness constraints. He told me in his usual friendly manner: This illness will kill me, while continuing life as usual, loving his family and his math, the latter much the same way as Bertrand Russell [25] did:

Mathematics, rightly viewed, possesses not only truth, but supreme beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Herb himself was neither cold nor austere, but radiated a warm and a gentle and a special Herb light. His contributions and meteor will continue to live, spread and shine for generations to come.



Photo #5: Aviezri Fraenkel and Herb Wilf, May 2010



## *Doron Zeilberger*

It is hard for me to write something solicited about W. He was the better half of “WZ pair” and “WZ theory”. If I had to choose one word to describe him, it would be “intertwined”. His beloved wife of sixty years, Ruth Tumen W., said in her eulogy, “for him research and teaching were intertwined”. I wholeheartedly agree, but not only teaching and research. The computational and the conceptual, the pure and applied, the big picture and the nitty-gritty details of Fortran or Mathematica code, research and exposition (and boy was he a great expositor!) the professional and the personal, and so on and so forth.

He was also way ahead of his time! When computers were just starting to get used, he edited, in the late fifties, together with Anthony Ralston, a seminal “cookbook” telling people how to solve mathematical problems with computers, complete with flowcharts and pseudocode (before Fortran!). Fifteen years later, he co-wrote, with Albert Nijenhuis, the bible of combinatorial algorithms, which intertwined theory and implementation. When the internet started, he pioneered the *Electronic Journal of Combinatorics*, a free(!) on-line journal. He convinced the publishers of his numerous books to let them post his books freely on the internet, thereby being a pioneer of open-access, and open source-code. When Tim Gowers asked him to write the entry for the *Princeton Companion on Mathematics* on “Enumerative and Algebraic Combinatorics” he politely refused, but suggested that he write instead an entry on “Mathematics: an Experimental Science”, thereby introducing this avant-garde subject into the canon of mathematics.

And his mathematics was divine! See also [www.math.rutgers.edu/~zeilberg/](http://www.math.rutgers.edu/~zeilberg/) for lectures and talks, especially the lecture “Herb Wilf, in memoriam” Feb. 2, 2012 .



Photo #6: Doron Zeilberger, Herb Wilf, Gainesville, 2005.

## *Marko Petkovšek*

I first got in touch with Herb while working on my thesis in Computer Science at Carnegie Mellon University under the supervision of Dana Scott, in the first half of 1990. By then, I had implemented a package for solving recurrences by the method of generating functions in *Mathematica*, which at the time was just two years old. My plan was to design an algorithm for finding Liouvillian solutions of linear difference equations with polynomial coefficients, in analogy to the well-known Kovacic/Singer algorithm in the differential case. I realized that one should work with difference rings rather than fields, but was unable to develop the requisite Galois theory (Singer and van der Put did that in 1997). So I tried the “small steps” approach. First I looked for polynomial solutions, which was not too difficult. Then I tried to find rational solutions, but



could not get a suitable denominator. I was not aware that Sergei Abramov had solved this problem in 1989. But I had just learned about Gosper's algorithm and hypergeometric terms from the newly published *Concrete Mathematics* by Graham, Knuth and Patashnik, pointed out to me by my fellow graduate student Todd Wilson. I managed to come up with an algorithm for finding hypergeometric solutions, but was still in the depths of despair: My original goal seemed just as remote as when I started; I felt that I had no good results to put in my thesis; and my time was running out since I had to return home by October that year.

Luckily for me, Herb's *generatingfunctionology* appeared in 1990 as well, and my advisor presented me with a copy fresh from the press. At the end of the preface, Herb invites the readers to send him corrections. After reading the whole book avidly, I did so. As an afterthought, I mentioned in my message that I had an algorithm for finding hypergeometric solutions. Immediately I got a most enthusiastic reply, which changed my life: Herb explained to me that this was just the missing piece of the puzzle that he and Doron Zeilberger needed to solve the old open problem of whether a definite hypergeometric sum could be expressed in closed form. In short, without knowing it, I already had a very good result, and I was saved! But Herb did not satisfy himself with that: He wrote a circular letter to George Andrews, Richard Askey, Donald Knuth, and Dana Scott, telling them what a great thing I had done. My advisor was, of course, happy about this turn of events, too. At my defense in September, before any questions were posed, he stood up and read Herb's letter out loud. After that, the question period was a breeze for me . . . Herb also invited me to visit the University of Pennsylvania and give a talk there after my defense in Pittsburgh. He said that he would invite Doron as well, adding that he was not sure if Doron would come. But Doron came, and after my talk he posed me two questions. I don't remember what they were, but I do remember that they were right to the point, and that I was amazed at how insightful Doron was. This was the first time that I met Herb and Doron in person.

The second time that Herb affected my career was in September 1993 at the *Symbolic Computation in Combinatorics*  $\Delta_1$  meeting at Cornell University. One day during dinner at the famous Moosewood Restaurant in Ithaca, NY, Herb took me and Doron aside and proposed that we write a book about automated proofs of identities involving hypergeometric sums. We agreed, and Herb drafted the table of contents for nine chapters, which we divided among ourselves. In two years, Doron and I wrote two chapters each, and Herb wrote the remaining five chapters. I would have kept working on the draft forever, correcting errors and polishing the details, but Herb urged us to hurry because things were "in the air," and there was the possibility that others might overtake us. He selected the publisher – Alice and Klaus Peters, his good friends. Herb prepared a list of 50 or so possible titles for the book, one wittier than the next. In the end, we settled on the eye-catching  $A = B$ . The book was published in December 1995. Although I am listed as the first author because Herb and Doron insisted on alphabetical order, Herb was without a doubt the principal author.

The third time that Herb made a difference in my life was in 2006 when I was in the middle of my two-year stint as department head in Ljubljana. I was desperate about how I was going to survive it, and he offered me the chance to "recuperate" on a sabbatical afterwards. So I spent the spring term of 2008 as a visiting scholar at the University of Pennsylvania, which helped me get my research back on track. But most of all, Herb inspired me with his unfailingly positive attitude, his wit and good humor, his warm personality, his kindness and care for his students and coworkers. When I was doubting my mathematical abilities, he gave me his paper *Self-esteem in Mathematicians* [20] intended for young mathematicians with just this kind of insecurity, and it

helped me a great deal. To put it simply, he was the rare kind of person who makes the people around him happy. He was a great mathematician and could have behaved like a *prima donna*, but he never did. Just the opposite – he was a wonderful human being, uncomplaining and unselfish, sensitive to the needs and feelings of others, and that is how he lives on in my memory.

## *Neil Calkin*

Some memories of my friend and mentor, Herbert S. Wilf: I first met Herb at a conference on the west coast, I believe in 1985 when I was still a beginning graduate student. He was incredibly generous with his time and attention, and especially with his ideas.

The next time I met him, a couple of years later, he remembered me, and what sorts of things I was interested in, and took care to pose some questions he thought might interest me. Over the next few years we met up at conferences often, and would talk, and developed (for me, I feel, at least) a close friendship.

In 1993, at Herb's former postdoc Jamie Radcliffe's wedding, I learned of the World Wide Web, and of the beta release of NCSA's Mosaic web browser. Consequently, the following January, when Herb suggested that we needed a mailing list in combinatorics, I was ready to suggest that the time was right to start an on-line journal in combinatorics, a much more ambitious project. The very same day (I don't know whether it was before or after I talked to him) Fan Chung also mentioned Mosaic to him, and so the project seemed like it might be interesting.

Herb was travelling to a meeting in Bordeaux a few days later for a two-week trip, and said he'd consider the idea upon his return. So I got to work learning how to set up a web server, write HTML and CGI scripts, and put together a mock-up of the journal, including fake papers such as "On the structure of the empty set", by Neil J. Calkin and Herbert S. Wilf, and "On the number of subsets of the empty set", by Herbert S. Wilf and Neil J. Calkin.

On seeing the mock-up, Herb immediately recognised that the technology was mature enough to realize our vision, and so he set about the much harder task of assembling a stellar editorial board and twisting arms to get some excellent papers for the first volume of the *Electronic Journal of Combinatorics (E-JC)*. We announced the journal to the world a mere three months later, in April 1994, and it has gone from strength to strength since then.

For a while we were seen as leaders in the new field of electronic publishing in mathematics, and were invited to various gatherings to discuss our endeavours. At one meeting, at MSRI, I referred to the E-JC as being like "my child", and I remember how upset he was at the time: Ever after, I referred to it as "our child!"

There was a great deal of skepticism among some in the publishing community that E-JC could survive: People told us we didn't have a sustainable model, that it was bound to fail for lack of interest, that people wouldn't take it seriously, and most worryingly, that once Herb and I were not involved, it would fall apart. At least partly with this in mind, we both made efforts to bring other people into the fold, and to step back when the time was right, so that others could demonstrate that the E-JC had an existence beyond either one of us. And I think that, looking at how strong the journal is now, we have proved that point!

Herb and I wrote three papers together. It could have been many more, had we been in closer physical proximity — for all my love of electronic communication, I still work better in

the same room with my collaborators. That said, working with Herb over email was a delight: bouncing ideas back and forth until we'd obtain a beautiful result — or discovering that we've just reinvented something in Chapter 4 of Comtet's *Combinatorial Analysis*.

Our first paper, "On counting independent sets in grid graphs", remains one of my most cited papers: It was the convergence of work that each of us was doing independently (in my case, building on work I'd done some years earlier in my thesis; in his, building upon a paper on counting non-attacking configurations of kings). It's apparently of great interest in the IEEE community, and as the results are not yet sharp, it's a problem I still return to frequently.

Our second paper, "Recounting the rationals", has garnered more attention for me than anything else I've been involved with. I've had more than twice as many emails and comments about that paper than on all my other work combined. There is a Wikipedia page about the "Calkin-Wilf Tree" (which we didn't name!) and one can even buy a t-shirt with a wonderful picture of the tree and rendition of our proof.

I learned a very important lesson on that paper: not to give up! When we submitted the paper, a referee pointed to a paper in a birthday conference proceedings which proved some of the results we had proved: Naturally we withdrew the paper. Herb, however, recognised that although the countability of the rationals was incredibly well known, it was worth writing up our proof of that, and so the note in the *Monthly* came about. I tell all my students this story as a life lesson!

Our final paper together, "On a shooting game of Lampert and Slater", with Rod Canfield, was a delight to research, and a joy to write. It showed Herb's touch and grace: When we couldn't analyze the original question, he suggested a simpler model, which we *were* able to analyze exactly. This led us first to being able to understand what the behavior of the shooting game would be, and also to develop a technique to prove it, incidentally, requiring fairly heavy computation along the way, making it very much of the flavor of experimental mathematics.

Herb and I would frequently send emails back and forth about various topics in mathematics: I remember one long exchange in which I was developing all sorts of ideas, with Herb sending encouragement after encouragement after encouragement: until finally he realized that I had recreated Faa Di Bruno's not-well-enough known formula for the derivative of a function of a function. He made sure that I knew not to be disheartened by this: to understand that to rediscover something gorgeous is good in and of itself. This was typical of his generosity of spirit and nurturing manner.

Herb was tremendously supportive of me in all ways, not least by writing letters of reference and recommendation for me for jobs and (I believe) tenure and promotion. He was a wonderful mentor to me, an inspiration mathematically and pedagogically, and a great man. I am very fortunate to have called him my friend, and feel incredibly honored that he called me friend in return. He is, and will remain, sorely missed.



Photo #7: Herb Wilf, Boca Raton, 2002.

## *Rodney Canfield*

I treasure time spent one-on-one with Herb, usually at conferences, visiting and chatting. Mostly about math, of course, but over the years I learned a little about his interesting personal history. He told me about living with his Aunt Martha while his parents were overseas for a while, and about going off to MIT as an undergraduate. He painted a picture of a rather naive young man, but one who had many adventures.

Even before graduate school at Columbia, Herb began his acquaintanceship with computers. The delightful story of how in 1951 he came to be employed by Nuclear Development Associates, while searching for duplicate bridge opportunities, is told on his website in an essay honoring his mentor, Gerald Goertzel. (Also among his fellow programmers was Alicia Nash.) Throughout his career Herb was enthusiastic about new ways to use computers in mathematics. He was among the first I knew to use  $\text{\TeX}$ , or email, or to embrace the World Wide Web. Regarding the latter, in 1994, in collaboration with Neil Calkin, he founded an online journal, the *Electronic Journal of Combinatorics*. Then, and now, in addition to being delivered via the WWW, the journal adheres to the dual goals of being cost-free to readers and authors, and of leaving copyrights with the authors. Presently, the future of scientific publication and the role of commercial publishing companies therein is very much in the limelight. (See the June/July, 2012, issue of the Notices, for example.) In the pre-electronic publishing days (1979) Herb was also the founding editor, along with Donald Knuth, of the *Journal of Algorithms*.

My first personal contact with Herb came when he was the editor of the *Monthly*, in the form of a request for a referee's report. Although I had a lot to say about the paper, I lacked the confidence to say it, and kept putting the report off. I got a stern reminder that it was due, and so I didn't know how else to proceed except to write up what I had been thinking. I got back a prompt thank-you note saying, "Well worth waiting for," and it was most gratifying. Shortly after this, thanks to Ed Bender's friendship with Herb, I had the chance to attend an informal math workshop at Herb's beach house on the New Jersey shore. Besides Ed and myself, Bob Robinson, Bruce Richmond, and Nick Wormald were in attendance. We did math endlessly, and it is a truly fond memory.

For about the last twenty-five years I have had fairly regular correspondence with Herb. Anytime I had a question, or wanted an opinion, I could write and receive an immediate useful reply. I tried not to overdo my welcome, and also to reciprocate by being a good correspondent. It was not so hard for me, because I did not have people all over the world writing to ask me my opinion, as I am sure Herb did. I suspect he was just as prompt with virtually all inquiries, and I am mystified as to how he did it. Part of the answer is that he had such a fine mind, and the other part is a testimony to his dedication.

When it comes to talking about Herb's accomplishments in research, it is literally impossible to know where to start. Surely one of the most lasting and significant breakthroughs was his work with Doron Zeilberger on the  $WZ$  pairs. For me personally certain other of Herb's interests are especially appreciated. His original book with A. Nijenhuis on *Combinatorial Algorithms*, published in 1978, is one of my prized possessions. I have spent an incredible amount of time with that book. The various problems associated with efficient listing, and navigating through combinatorial objects, I think, are totally absorbing. Besides this work on combinatorial algorithms, Herb has authored an impressive number of other influential books, including works on identities, algorithms and complexity, numerical analysis, mathematical methods for the physical sciences, integer partitions, and generating functions. I'd like to say something about the last.

The generating function, historically as old at least as Laplace, is one of the most powerful and dependable tools in combinatorics. As a graduate student I soaked up the tricks and techniques of generating functions. But later in an undergraduate course my students were perplexed by the topic. When they asked for a good reference on the topic, I had no ready response. Until, that is, 1990, when the first edition of Herb's *generatingfunctionology* appeared. A beautifully written, appropriately titled, and totally useful, text, it is a perfect starting place for ambitious students, and well worth the occasional visit from the expert.

In the last chapter of *generatingfunctionology* Herb comes to analytic and asymptotic methods. In the preface he likens using generating functions only in the formal sense without analysis to listening to a symphony in stereo but with only one channel turned on. Given his mathematical roots, it is no surprise that as Herb turned his attention to combinatorial problems he brought along an abundance of analytic techniques. Having myself been once congratulated on having the only contour integral at a combinatorics conference, this love of analytic methods was a common bond for us. In 1990, when *generatingfunctionology* appeared, it had been twenty-two years since Herb published the paper "A mechanical counting method and combinatorial applications." In that paper he combined Cauchy's formula for coefficient extraction with inclusion-exclusion. In his review of the paper, David Klarner wrote, in part, "Using these results the author (a) describes a formula for the number of Hamilton circuits in a connected graph involving about  $n^4 2^n$  operations, (b) finds the number of 1-factors of a graph, (c) gives a new formula for the

permanent of a square matrix and derives Ryser's formula, (d) solves the generalized 'probl  m des m  nages,' and (e) gives inequalities for the permanent function which are superior in some cases to the old bounds. It should be clear from this impressive list that the elegant generating function technique developed in this paper is worthy of further study."

Our favorite topic was integer partitions. We often wrote and talked about these, and have worked on a number of problems. Here is one. Notice that if you allow yourself the binary powers  $1, 2, 4, \dots$  as parts, and only 0 and 1 as multiplicities, then there is exactly one partition of each positive integer  $n$ . **Problem.** Do there exist an infinite set of parts  $A = \{1 \leq a_1 < a_2 < \dots\}$  and an infinite set of multiplicities  $M = \{0 = \mu_1 < \mu_2 < \dots\}$  such that  $p(n; A, M)$ , the number of partitions of  $n$  whose parts belong to  $A$  and whose multiplicities belong to  $M$ , satisfies:

- $p(n; A, M) > 0$  for all  $n$  sufficiently large
- there exists constant  $C$  such that

$$p(n; A, M) = O(n^C), \quad n \rightarrow \infty.$$

We were unable to resolve this question, and it is listed as unsolved problem #2 on Herb's website. Recently, Noga Alon has given a complete solution, see *Integers*, Volume 13, 2013.

Besides his monumental body of work, and his generous service to the profession, Herb had a profound effect on many younger persons who love mathematics. He posed intriguing problems of just the right level of difficulty. He was a great source of encouragement, and we are all left with a deep sense of personal loss at his passing.

## *Albert Nijenhuis*

When Herb Wilf, then a junior faculty member at U. Illinois, came to Penn in the spring of 1962 as part of the year-long Hans Rademacher retirement conference, I. J. Schoenberg, the department chairman, was so impressed by this young man that, in consultation with the department and the university dean, he offered him an Associate Professorship, with tenure. Herb accepted, with pleasure, looking forward to an active relationship.

Things worked out differently. Within a year, Schoenberg moved elsewhere. Herb continued working on his other numerous interests, but was looking for another focus.

In 1964 I joined the Penn Math Dept, still actively involved with deformation theory, but I sensed this was approaching its end. I was also looking for a new focus and had had occasional contact with Herb.

Around that time, the faculty and students at Penn gained access to the mainframe computer on campus. The procedure was clumsy, but Herb and I were both interested. Herb had previous experience with computers: In his graduate student days he had participated in a project simulating the operation of a nuclear reactor. At that time machine language was used; now FORTRAN was available. Here we found an area of mutual interest. Herb was the first to teach a section of Freshman Calculus that required the use of a computer; others followed. He also wrote an introductory booklet on FORTRAN, less detailed but more accessible to beginners than standard texts. We discovered how suitable computers are for the exploration of combinatorial structures, as well as number-theoretical questions. Herb decided to become a combinatorialist

and spent much of 1966-67, while on leave at the London Imperial College, acquainting himself with the field.

During the year 1973-74 Herb was on leave at Rockefeller University, on a Guggenheim Fellowship. When he returned to Penn, he brought with him an outline for a book. It was an early version of what would be our book, *Combinatorial Algorithms*.

Each chapter would contain a mathematical discussion, followed by an algorithm and a FORTRAN program. The book was therefore both a textbook suitable for a combinatorics course and also a reference for immediate application.

The chapters that deal with “combinatorial families” have a novel feature. Each time the algorithm is called, it chooses another member of the family. The sequencing rule for the choices depends on the nature of the family. One option is uniform randomness; another interesting one is the so-called “revolving door” method.

In spite of some mixed reviews the book was well received. The first edition in 1975 was followed by the second, in 1978. Gradually, FORTRAN went out of style, and we did not envision another edition. However, when Herb made a number of his books freely available on-line, this book was among them, and from then on it has enjoyed unexpectedly large numbers of “hits”.

During the 20-or-so years together at Penn, there was steady traffic between our respective offices, resulting in published papers and chapters in the book. At the same time, Herb kept up with his other interests, resulting in papers on a variety of subjects, with a variety of co-authors. His teaching at Penn earned him a distinguished campus-wide award, and his guest-lecturing a national MAA award. The latter activity was aided by Herb’s private airplane, which provided his preferred mode of travel.

Herb had many hobbies, often shared with his wife Ruth. These included flying, birding, boating, photography, gadgets, . . . He and Ruth created a warm and hospitable environment as they informally entertained their friends at their shore house in New Jersey. Always present: Herb’s lightning-quick sense of humor.



Photo #8: Albert Nijenhuis, Herb Wilf, 1996.



## *Carla D. Savage*

I first met Herbert Wilf in 1988 at the SIAM Conference on Discrete Mathematics in San Francisco. He gave a spectacular invited address about Combinatorial Algorithms, the subject of his 1989 SIAM Monograph.

He posed several intriguing questions about combinatorial Gray codes. I solved one and wrote to him about it. Herb was enthusiastically encouraging and invited me to the University of Pennsylvania to give a talk.

What is so special about Herb is that the previous paragraph could have been written about him by any one of the dozens of mathematicians he has inspired and encouraged through the years.

I accepted Herb's invitation to give a talk and we started an email correspondence about problems and results. Herb's wit is legendary, so, as you can imagine, those emails are full of gems. He put me in touch with Gray code expert Frank Ruskey, my collaborator on several interesting projects through the years.

Herb Wilf was a great problem-solver who made many contributions to mathematics. But he also loved learning and communicating about mathematics. I think one of the things he most enjoyed about his work with the *Journal of Algorithms*, the *American Mathematical Monthly*, and the *Electronic Journal of Combinatorics*, was getting to be the first one to hear about new results. And he was masterful at explaining hard mathematical ideas inside of books with playful titles, like *generatingfunctionology* and  $A=B$ , with Petkovšek and Zeilberger.

Our first collaboration was not until 1996. Herb suggested some questions about integer partitions and he allowed me to include a few students in our discussions (one of them was Sylvie Corteel). They made some great discoveries and became our co-authors [16, 17]. During this time, Herb flew his plane down to N.C. State to give a talk and meet with the group, a memorable experience for us all.

Herb asked unusual questions about familiar mathematical objects and started entirely new lines of inquiry. Here are a few examples from the realm of integer partitions.

- Herb posed this question in his talk at the 1988 SIAM Discrete Math Conference: Is it always possible to list the partitions of integer  $n$  in such a way that an element differs from its predecessor on the list only in that one part has increased by 1 and one part has decreased by 1? (A part of size 1 may decrease to 0 or a part of size 0 may increase to 1.) For example, the following list of partitions of  $n = 6$  satisfies the requirements:

1 1 1 1 1 1, 2 1 1 1 1, 3 1 1 1, 2 2 1 1, 2 2 2, 3 2 1, 3 3, 4 2, 4 1 1.

Surprisingly, the answer is yes [29].

- Herb asked: What is the probability that a random partition of an even integer  $2n$  is the degree sequence of a simple graph? Several tried their hand, including Erdős and Richmond [27]. Finally, Pittel showed that the probability goes to 0 as  $n \rightarrow \infty$  [28].
- A partition of  $n$  can be represented by its Ferrers diagram, consisting of  $n$  unit squares. Herb asked if, for large enough  $n$ , an  $n \times p(n)$  rectangle could be tiled by the Ferrers diagrams corresponding to the  $p(n)$  partitions of  $n$ . Alon, Bona, and Spencer [26] were able to show it is not.

- Herb considered the following: As  $n \rightarrow \infty$ , what is the probability that a randomly chosen part size in a random partition of  $n$  has multiplicity  $m$ ? It turns out that the probability approaches  $1/(m(m+1))$  as  $n \rightarrow \infty$  [15].
- Consider partitions  $x_1 + x_2 + \cdots + x_k$  of  $n$  where  $x_1 \geq x_2 \geq \cdots x_k > 0$ . Given nonnegative integer  $m$ , for how many partitions of  $n$  is  $m = x_2 + x_4 + x_6 + \cdots$ ? Herb observed that for  $n \geq 3m$ , there is a very nice answer and it is independent of  $n$  (!) We were able to explain and generalize this in [14] with Rod Canfield.

One of the side benefits of collaborating with Herb was getting to know his wife, Ruth, a renowned midwife and childbirth educator, and organizer extraordinaire, who is now my own friend. I am grateful to Herb and Ruth for their hospitality during my many visits to Philadelphia. I am full of memories of joyful meals with them at conferences all over the world, talking about math, life, and childbirth. Some standouts: Philadelphia (food trucks, cheesesteak subs), the Southeastern Conference in Boca Raton (lunch with Sister Celine), MADCAP in Baltimore (steamed crabs), the Clemson Mini-Conference (he flew me there in his plane!), Herb's PIMS lectures on integer partitions (see <http://www.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf>) in Victoria (Butchart Gardens, more steamed crabs), FPSAC in Sicily (gelato), CanaDAM in Montreal (bagels), and Chapel Hill (Ruth's 80th birthday dinner).

Our last paper together was begun in March 2011 when Herb and Ruth were in Chapel Hill for a childbirth conference that Ruth was chairing. While Ruth attended the conference, Herb and I sat at a table in the lobby talking about hypergeometric identities associated with binary words. Herb wrote to George Andrews about it and with George's response, our joint paper was born [13].

With Herb, doing mathematics was not only exciting, but, sometimes, hilarious. I will greatly miss sharing it with him.



*Photo #9: Back row: David Wilf, Peter Wilf, Eugene Zima (conference co-organizer), Ruth Wilf, Susan Wilf, at the W80 birthday celebration, Waterloo, 2011.*

## Richard P. Stanley

One of Herb Wilf's most influential contributions to mathematics is the theory of pattern avoidance. The first result in this area is due to P. A. MacMahon [36, §§96–97], who showed that the number of permutations  $w \in \mathfrak{S}_n$  (the symmetric group of all permutations of  $1, 2, \dots, n$ ) with no increasing subsequence of length three is equal to the Catalan number  $\frac{1}{n+1} \binom{2n}{n}$ . Like almost all of MacMahon's work, this striking result was forgotten or ignored until the combinatorial renaissance of the 1960's. MacMahon's result was rediscovered by Hammersley [33] around 1970. Hammersley's result arose from a famous problem posed by Ulam [40], asking for the expected length of the longest increasing subsequence of a random permutation in  $\mathfrak{S}_n$ . Meanwhile Donald Knuth in the 1960's considered the question of sorting a permutation  $w = a_1 a_2 \cdots a_n$  on a stack. We can define recursively the result  $S(w)$  of passing  $w$  through a stack as follows. Write  $w = unv$ , so  $u$  and  $v$  are permutations of complementary subsets of  $\{1, 2, \dots, n-1\}$ . Then  $S(w) = S(u)S(v)n$ . Knuth [Exercise 2.2.1.5 in *The Art of Computer Programming*, Vol. 1] showed that  $w$  is stack-sortable, i.e.,  $S(w) = 12 \cdots n$ , if and only if  $w$  is 231-avoiding, i.e., there does not exist a subsequence of  $w$  of length three whose terms are in the same relative order as 231 (smallest term last, second-smallest term first, and largest term in the middle). Equivalently, there does not exist  $i < j < k$  with  $a_k < a_i < a_j$ . Knuth [Exercise 2.2.1.4 in *The Art of Computer Programming*, Vol. 1] also showed that the number of 231-avoiding permutations  $w \in \mathfrak{S}_n$  is again the Catalan number  $C_n$ .

With the benefit of hindsight it seems pretty obvious to consider permutations avoiding a permutation  $v \in \mathfrak{S}_k$  for any  $k \geq 3$ , or more generally, avoiding a set of permutations in various  $\mathfrak{S}_k$ 's. Thus for instance 3614725 does *not* avoid 2143, since the subsequence 3175 has its elements in the same relative order as 2143. The first person to come up with this "obvious" insight was Herb Wilf. Previously pattern-avoiding permutations arose in connection with other problems, but no one thought of investigating these permutations for their own sake. The first paper [38] on general pattern avoidance *per se* was published in 1985 by Herb's Ph.D. student Rodica Simion, in collaboration with Frank Schmidt. They considered even and odd permutations avoiding sets of permutations in  $\mathfrak{S}_3$ .

Write  $s_n(v)$  for the number of permutations in  $\mathfrak{S}_n$  avoiding the permutation  $v$ . In unpublished work in the 1980's, Wilf raised the question of when two permutations  $u, v \in \mathfrak{S}_k$  satisfy  $s_n(u) = s_n(v)$  for all  $n \geq 1$ . These permutations are now called *Wilf-equivalent*, and the equivalence classes are called *Wilf classes*. Probably the first paper to use the term "Wilf-equivalent" (in fact, in the title) is by Babson and West [32]. Thus for instance all permutations in  $\mathfrak{S}_3$  are Wilf-equivalent, while there are three Wilf classes in  $\mathfrak{S}_4$ . A great deal of work has gone into the theory of Wilf equivalence, but a definitive understanding is lacking.

It is a simple consequence of the RSK-algorithm that

$$\lim_{n \rightarrow \infty} s_n(1, 2, \dots, k)^{1/n} = (k-1)^2.$$

(A much stronger result was proved by Regev [37].) Herb and I (independently) asked around 1980 about the behavior of  $s_n(v)$  for general  $v \in \mathfrak{S}_k$  and large  $n$ . Herb first asked (when unaware of the work of Regev) whether  $s_n(v) < (k+1)^n$  for all  $v \in \mathfrak{S}_k$ , while I asked whether  $\lim_{n \rightarrow \infty} s_n(v)^{1/n} = (k-1)^2$  for all  $v \in \mathfrak{S}_k$ . Herb quickly modified his original conjecture by asking whether there exists a number  $L_v < \infty$  such that  $\lim_{n \rightarrow \infty} s_n(v)^{1/n} = L_v$ . This modified

conjecture was first published by him in [6] well after first circulating the problem. Meanwhile Miklos Bóna and others asked the seemingly weaker question of whether  $s_n(v) < c^n$  for some constant  $c = c(v)$ . Arratia [31] proved that the existence of  $L_v$  is equivalent to the inequality  $s_n(v) < c^n$ . An affirmative answer to these two equivalent questions became known as the *Stanley-Wilf conjecture*, a name coined by Bóna. It is interesting that Wilf expressed a little doubt about the conjecture because Alon and Friedgut [30] found an upper bound  $c^{n\alpha(n)}$ , where  $\alpha(n)$  is the inverse Ackermann function. The conjecture was finally given an exceptionally elegant proof by A. Marcus and G. Tardos [21] in 2004. As in the theory of Wilf classes, there are many open problems related to the Stanley-Wilf conjecture. For instance, are the numbers  $L_v$  always algebraic? This seems highly unlikely but remains open. For what  $u, v \in \mathfrak{S}_k$  do we have  $L_u < L_v$  or  $L_u = L_v$ ? Somewhat surprisingly, the largest and smallest values of  $L_v$  for  $v \in \mathfrak{S}_k$  are not achieved by the identity permutation  $v = 12 \cdots k$ .

Herb had two papers in addition to [6] related to pattern avoidance, both of which anticipated future developments. The first, with Emeric Deutsch and A. J. Hildebrand [18], is devoted to the distribution of the longest increasing subsequence in certain pattern-restricted permutations. The second paper, with Carla Savage [19], concerns generalizing pattern avoidance from permutations to compositions and multiset permutations.

The theory of pattern avoidance is now a flourishing subject in its own right, with a vast number of variations and applications. As a sample application, Lakshmibai and Sandhya [35] show that a Schubert variety  $X(w)$  in the complete flag variety  $GL(n, \mathbb{C})/B$  (where  $B$  is a Borel subgroup) is smooth if and only if  $w$  avoids 3412 and 4231. Further applications of pattern avoidance may be found in Bridget Tenner's database [39]. Google gives over 1,200,000 hits for the search "pattern avoidance Wilf." An annual conference on Permutation Patterns was begun in 2003, and a book on patterns in permutations and words by Sergey Kitaev [34] was published in 2011. The pattern avoidance seedlings planted by Herb Wilf have developed into a proliferant and fecund forest.



Photo #10: Herb Wilf, Paul Erdős, 1996. (Photo by Albert Nijenhuis)

## *Felix Lazebnik*

Herb Wilf was my Ph.D. advisor at the University of Pennsylvania from 1984 until 1987. The main reason I asked him to be my advisor was that I liked very much a two-semester course in

Combinatorics that I took from him in 1983-1984. The further that course progressed, the more impatient I became waiting for the next lecture.

Much later I tried to analyze what exactly he did in the course that was so stimulating. He lectured calmly and smoothly, and if ever a difficulty arose, he resolved it quickly, either immediately or in the next lecture. I remember a day when no one could solve an assigned homework problem. When we told him he looked surprised and said he was really sorry about this. That was strange – wasn't it customary to challenge students with hard problems? Furthermore, Herb did not give broad panoramic views and motivation, nor did he mention history – features that I valued in courses. So why did I like his course so much?

Here is one reason I can name: It was not just combinatorics, it was mathematics. Simple analysis, operators involving partial derivatives, light number theory, or linear algebra were used without warnings. A long transformation was carried out on the board and was not left for a dull homework problem. Some needed facts (like Euler's summation formula, or the Gauss-Lucas theorem, etc.) were not mentioned as 'well known', but presented and proved. And it was clear that Herb *enjoyed* talking about this mathematics to us. It was not an onerous duty for him, it was sharing of love. Another characteristic of his lectures was his ability to present hard proofs in a such clear way that a listener was left with an impression that nothing extraordinary was happening. Until one tried to reconstruct these proofs.

Later, when I took a reading course with him, Herb asked me to read and do problems (selected by him) from D.E. Knuth's *The Art of Computer Programming, Vol. 1* and L. Lovász's *Combinatorial Problems and Exercises*, both books with helpful solutions included. He asked me whether I did what he assigned, whether I had questions, and then assigned new reading and new problems. During the first year of my research, Herb wanted to see me regularly once a week. Often I had very little to report, and so felt less than enthusiastic going to these meetings. When on those days I was leaving his office, I felt great. My spirit was up, and I was full of energy and optimism for another week. What he said or did for this – I have no idea! All our meeting rarely lasted more than 15 minutes, and usually took place late afternoon. His parting words were typically "peace, enjoy . . .", with a hand raised slightly and a smile on his face.

When Herb agreed to be my advisor, he said that his task was to suggest a good problem, and that he expected me to be independent. Later, his most common answer to my questions was a simple "I do not know", and, somehow, it was comforting. I believe that he did help me to become professionally independent.

Much later I tried to convince him to write at length about his teaching philosophy and methods, since they resonated well with so many of his students. His answer was a short "No", with a remark that teaching was an art and could not be replicated. Having taught for many years, I agree. I also agree with his view that "*... teaching is mostly a chemical activity that goes on between Human A and Human B.*" More of Herb's views on teaching can be found on his Penn website.

Herb was very reserved with giving unsolicited pieces of advice. I remember only this one: Never work on a problem you do not like. Here is one of Herb's problems (and also Linial's [42]) that I like and have worked on. It is about thirty years old, and is still a subject of active research. *What is the greatest number of proper vertex colorings in  $\lambda$  colors that a graph with  $n$  vertices and  $m$  edges can have?* I obtained some initial results [41], more recently good progress has been made (see for example [43], [44], [45]), yet the problem remains unsolved.

Doron Zeilberger once said that if he had to characterize Herb using one word only, it would be

*elegance*. I agree. (See also D. Zeilberger's contribution to realize that one word could be chosen in more than one way.) Restraint and grace of style – this was Herb. And humor. Attending his 80th birthday conference, together with others, I tried to help pushing the wheelchair. He said with a smile: "Well, I used to push my students, and now they are pushing me!"

We became closer during the years. Ruth and Herb have always been adventurous with food. The last dinner my wife Luba and I had with them was at *Uzbekistan* – a restaurant in far North East Philadelphia, and this was their idea. Four months before his death, Herb and Ruth invited us to visit them in their vacation home on Jersey shore. Something made us reschedule it, but then it was too late. We will always regret it.



*Photo #11: Herb Wilf and Fan Chung at the 65th Birthday Conference, 1996*





Photo #12: Backrow: Ellen Gethner, Carsten Thomassen, Ralph Stanton, Joan Hutchinson, Roy Levow, Ruth Wilf, Lis Thomassen Frontrow: Debra Boutin, Michael Albertson, Herb Wilf, Esther Levow, after the Euler Medal award ceremony, Boca Raton, 2002.

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