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Current; Forest Tree Theorem; Potential Functions and their Bounds

1 Introduction

In this section, we will continue our discussion on current and induced current. Review its properties and how it relates to spanning trees. Further, we will expand the Matrix Tree Theorem with a new Forest Tree Theorem which has many potential applications with current. Lastly, we will begin studying the

Recall that an electrical network can be viewed as a weighted graph, $G = (V, w)$ where w is a function $w : \binom{V}{2} \to [0, \infty)$. Specifically, $w(a, b) = 0$ when there is no edge $\{a, b\}$. It is important to note that the weight of an edge can have to meanings. It can refer to the distance between to nodes- i.e., ”how long the road is”, or it can refer to capacity, i.e., ”how wide the road is”. For electrical networks, it refers to the capacity or have much electricity can flow on that edge.

2 Current

Current is akin to network flow where one vertex is designated the source, s, and another the sink, t. The current obeys certain fundamental axioms. First, the current at the source, i_s is (without loss of generality) 1, at at the sink $i_t = -1$. Let i_V be the vector of the currents at each vertex and i_E be the vector for the current along each edge. (Note that for the purposes of current, each edge is oriented. To find the current in the opposite direction, multiply by -1) Also, the current around any closed loop is zero. Lastly, the current along each edge is given by $i_V = i_E\mathbf{B}$ (This is known as the Kirchhoff current law) where $\mathbf{B}$ is the $|E| \times |V|$ homology matrix associated with the combinatorial Laplacian, where $\mathbf{L} = \mathbf{BB}^*$.

Theorem 1. *On a connected electrical network, the axioms determine a unique value of* i_E

Proof. By hypothesis, G is connected, so the matrix $\mathbf{BB}^*$ has kernel of dimension 1. Hence, the matrix $\mathbf{B}$ has a left kernel of dimension of $|E| - |V| + 1$. If we look at a spanning tree of G, any omitted edges induces a cycle, and it is not hard to show that these cycles, together, form a basis for the cycle space of the graph. Hence, the dimension of the cycle space of G is $\mathbf{B}$ is $|E| - |V| + 1$. By the definition of $\mathbf{B}$, any cycle induces a current of 0. That is, the cycle space coincides with the left kernel of $\mathbf{B}$. Hence, the current of on the edges of a spanning tree determines the current for the entire graph and is unique. □

Aside. This technique of reducing problems to cycles and spanning trees is very useful. Moreover, by dualizing a planar graph, by considering the faces as vertices and the boundary edges as edges, we notice that cycles in the original graph are cuts in the dual and vice versa. Matroid Theory plays a prevailing role in this technique because it captures these concepts in easy-to-use spaces. However, the pitfall of Matroid Theory is that it often sacrifices adjacency for spaces, and hence, can be hard to use for general graphs.

Theorem 2. $i_{ab} = \frac{\tau(s,a,b,t)-\tau(s,b,a,t)}{\tau(G)}$ *where* $\tau(s,a,b,t)$ *is the number of spanning trees whose unique path from* s *to* t *(the source and sink) goes though* a *and* b *with* a *before* b*.*

Proof. See previous lecture. As an exercise the reader should compute a few i_{ab} for simple graphs. □

3 The Forest Tree Theorem

The Forest Tree Theorem is forest analog of the Matrix Tree Theorem proven in a previous lecture.

Recall that a *spanning thicket* is spanning thickets a 2-rooted, 2-component spanning forest such that s is in one component and t is in the other. The Forest Tree Theorem will count the number of spanning thickets using $\mathbf{B}$ and other tools used thus far.

Theorem 3. *Forest Tree Theorem*
Let $\mathbf{B}''$ *be the matrix* $\mathbf{B}$ *but deleting two rows- one corresponding to* s *and another to* t*, and likewise for* $\mathbf{B}''^{*}$ *and columns for* s *and* t*. Then,* $det(\mathbf{B}''\mathbf{B}''^{*})$ *is the number of spanning thickets (2-rooted, 2-component spanning forests such that* s *is in one component and* t *is in the other).*

Proof. Note that:

$$det(\mathbf{B}''\mathbf{B}''^{*}) = \sum_{S \subset E, |S|=|V|-2} det(\mathbf{B_S}\mathbf{B_S^*})$$

where $\mathbf{B_S}$ is the matrix $\mathbf{B}$ with only columns that correspond to S, deleting the others.

Observe $|det(\mathbf{B_S}\mathbf{B_S^*})|$ = 1 if S induces a thicket, and is 0 otherwise. This is because if s and t are on the same component, there is a non-zero vector, $\mathbf{v}$ corresponding to the unique path between them, that makes $\mathbf{B}''\mathbf{v}$ zero. And further, if there are more than 2 connected components, there must be a cycle whose corresponding edge vector is in the kernel of $\mathbf{B}''$.

The rest follows as in the Matrix Tree Theorem. □

It should be noted that, from a computational point of view, the determinant is easy to compute. And hence, the number of spanning trees are easy to compute. However on the other hand, the permanent, or the determinant ignoring sign changes from permutation matrices, which counts the number of matchings is hard to compute.

4 Potential Functions and their Bounds

Among the many topics in spectral graph theory, one of the most important concepts are spectral gaps. For the normalized Laplacian, all of the eigenvalues are between 0 and 2, and the gaps between certain eigenvalues, specifically, $\lambda_0 = 0$ and λ_1, the second-smallest eigenvalue, gives a lot of information about the graph.

Recall the Rayleigh Quotient:

$$\lambda_i = \inf_{f, f \perp \lambda_j, j<i} \frac{\sum_{x\ u} (f(x) - f(u))^2}{\sum_x f(x)^2 d_x}$$

In particular, for λ_1:

$$\lambda_1 = \inf_{f, \sum f(v)=0} \frac{\sum_{x \sim u} (f(x) - f(u))^2}{\sum_x f(x)^2 d_x}$$

Lemma 1. *Suppose f achieves the* inf *above. Then,* $\lambda_1 f(x) d_x = \sum_y f(x) - f(y)$

Proof. The First Proof

$$\lambda_1 = R(f) = \frac{< f, (\mathbf{D} - \mathbf{A}) f >}{< f, \mathbf{D} f >} = \frac{< g, \mathcal{L} g >}{g, g}$$

where $R(f)$ is the Rayleigh quotient, and $g = \mathbf{D}^{1/2} f$.
so

$$\lambda_1 g(x) = \mathcal{L} g(x)$$

with $g(x) = f(x) {d_x}^{1/2}$
then

$$\lambda_1 f(x) \sqrt{d_x} = g(x) - \sum_y \frac{g(y)}{\sqrt{d_x d_y}}$$

$$= \frac{1}{\sqrt{d_x}} \sum_{y, y \sim x} \frac{g(y)}{\sqrt{d_x}} - \frac{g(x)}{\sqrt{d_y}}$$

$$= \sum_{y, y \sim x} f(x) - f(y)$$

□

This is one technique for computing eigenvalues- simply manipulate the sums toward the desired result. Another proof is given by assuming a better f can be formed by perturbing f by ϵ and arriving at a contradiction

Proof. The Second Proof
Assume f can be perturbed by $\epsilon > 0$ to form f' such that $R(f') < R(f)$ as follows: Choose any basepoint x_0. let $f'(x_0) = f(x_0) + \epsilon/d_{x_0}$ and $f'(x) = f(x) - \epsilon/(vol(G) - d_{x_0})$ for $x \neq x_0$. Note that the new f' still meets the criteria that $\sum f'(x) = 0$. Hence,

$$0 \leq R(f') - R(f) = \varepsilon \frac{[\sum_y (f(x_0) - f(y))] - (\lambda_1 f(x_0) d_{x_0}) + O(\varepsilon^2)}{a\ sum\ of\ squares}$$

To complete the proof, which is left as an exercise to the reader, show that the numerator must be 0. □

Now we continue onto a theorem that places a bound on λ_1 in terms of $vol(G)$ and the diameter, $\mathbb{D}$.

Theorem 4. *If G is connected,*

$$\lambda_1 \geq \frac{1}{\mathbb{D} vol(G)}$$

Proof. Note $R(f) = \lambda_1$ and $|f(v_0)| = \max_v |f(v)|$

so

$$\lambda = R(f) = \frac{\sum_{x, x \sim y} (f(x) - f(y))^2}{\sum_x (f(x)^2}$$

Let P the shortest path from v_0 to another vertex whose length is $\mathbb{D}$

Then, continuing the equations above,

$$\geq \frac{\sum_{(x,y)\in P} (f(x) - f(y))^2}{|f(v_0)| vol(G)}$$

$$\geq \frac{\sum_{(v_{i-1},v_i)\in P} (f(v_i) - f(v_{i-1}))^2}{\mathbb{D}|f(v_0)|^2 vol(G)}$$

Now by applying the triangle inequality and a telescoping series, we see:

$$\geq \frac{|f(v_{\mathbb{D}}) - f(v_0)|^2}{\mathbb{D}|f(v_0)|^2 vol(G)} \geq \frac{1}{\mathbb{D} vol(G)}$$

□