In this section we provide alternate proofs for a few of the results presented in Spectral Graph Theory by Fan R. K. Chung. In particular we will prove the right half of Cheeger's Inequality $2h_G \geq \lambda_1 \geq h_G^2/2$, and then prove lower bounds for h_G and λ_1 in the case when G is edge-transitive, by using the probabilistic method.

In what follows $G = (V, E)$ is a connected finite simple graph. For reference we state the definition of the Cheeger constant.

$$h_G := \min_{S \subset V} \frac{E(S, \overline{S})}{\min(\text{Vol } S, \text{Vol } \overline{S})}$$

where $E(X, Y)$ is the number of edges $\{x, y\} \in E$ with $x \in X$ and $y \in Y$, and Vol $X = \sum_{x \in X} d_x$. By this definition, we know that we have to cut at least $h_G k$ edges to remove a subset of volume k from G (provided $k \leq \frac{1}{2}\text{Vol } G$). Thus h_G is a measure of G's overall resistance to being disconnected.

Theorem 1.

$$2h_G \geq \lambda_1 \geq h_G^2/2.$$

Proof. Here we provide an alternate proof only for the second half. (The first half is easier anyway). Let f be an eigenfunction for λ_1, so that

$$\lambda_1 = \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_u f^2(u) d_u}.$$

($\sum_{u \sim v}$ denotes a sum over *unordered* pairs $\{u, v\} \in E$). Next define $f_+ = \max(0, f)$ and $f_- = -\min(0, f)$. Then

$$\lambda_1 \geq \frac{\sum_{u \sim v}(f_+(u) - f_+(v))^2 + \sum_{u \sim v}(f_-(u) - f_-(v))^2}{\sum_u f_+^2(u) d_u + \sum_v f_-^2(v) d_v}.$$

Using the fact (for positive numbers) that $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$, this gives

$$\lambda_1 \geq \frac{\sum_{u \sim v}(f_+(u) - f_+(v))^2}{\sum_u f_+^2(u) d_u} \left(\frac{\sum_{u \sim v}(f_+(u) + f_+(v))^2}{\sum_{u \sim v}(f_+(u) + f_+(v))^2} \right)$$

(If not, replace f by $-f$). In the numerator use the Cauchy-Schwartz inequality, and in the denominator use $(a + b)^2 \leq 2(a^2 + b^2)$. We get

$$\lambda_1 \geq \frac{(\sum_{u \sim v} |f_+^2(u) - f_+^2(v)|)^2}{2(\sum_u f_+^2(u) d_u)^2}. \tag{1}$$

Next we'll work on the numerator of this expression. Order the vertices such that $f(v_1) \geq f(v_2) \geq \ldots \geq f(v_n)$, and set $S_i = \{v_1, v_2, \ldots, v_i\}$. Every edge runs between S_i and $\overline{S_i}$ for some unique i, so

$$\begin{aligned}\sum_{u\sim v}|f_+^2(u)-f_+^2(v)| &= \sum_{i=1}^{n-1}(f_+^2((v_i)-f_+^2(v_{i+1}))E(S_i,\overline{S_i}) \\ &\geq \sum_{i=1}^{n-1}(f_+^2((v_i)-f_+^2(v_{i+1}))h_G \text{Vol } S_i) \\ &= h_G\sum_{u}(f_+^2(u)d_u).\end{aligned}$$

Substituting this into the numerator of (1) gives $\lambda_1 \geq h_G^2/2$ as desired. □

Next we consider graphs with a high degree of symmetry. The automorphism group Aut G acts on vertices, edges, paths, etc. of G; we say G is *vertex-transitive* or *edge-transitive* if Aut G acts transitively on the corresponding set.

Theorem 2. *If G is edge-transitive with diameter D, then*

$$h_G \geq \frac{1}{2D} \tag{2}$$

$$\lambda_1 \geq \frac{1}{D^2} \tag{3}$$

Proof. Let P be a random path chosen in the following way. Choose a random vertex u with probability proportional to its degree; i.e. $\Pr(u=x)=\frac{d_x}{\text{Vol } G}$, and choose v independent of u with the same distribution. Now choose P uniformly at random among all shortest paths from u to v.

The length of P is the sum of indicators $\sum_e 1(e\in P)$. Taking the expectation gives

$$D \geq \sum_e \Pr(e\in P).$$

By edge transitivity the summands all equal a constant value. Therefore for any *fixed* e,

$$\Pr(e\in P) \leq \frac{D}{|E|} = \frac{2D}{\text{Vol } G}. \tag{4}$$

Now to prove (2) we let $S\subset V$ be arbitrary, although for convenience we assume Vol $S \leq \frac{1}{2}$Vol G. In the event that $u\in S$ and $v\in\overline{S}$ (or vice versa), P must contain an edge between S and $\overline{S}$. Therefore (4) gives

$$2\frac{(\text{Vol } S)(\text{Vol } \overline{S})}{(\text{Vol } G)^2} = \Pr(u\in S, v\in\overline{S} \text{ or } u\in\overline{S}, v\in S) \leq E(S,\overline{S})\frac{2D}{\text{Vol } G},$$

and so

$$\frac{E(S,\overline{S})}{\text{Vol } S} \geq \frac{\text{Vol } \overline{S}}{D\text{Vol } G} \geq \frac{1}{2D}.$$

We conclude $h_G \geq \frac{1}{2D}$, which is (1).

For (3) we use the formula

$$\lambda_1 = (\mathrm{Vol}\ G)\frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_{\{u,v\}}(f(u)-f(v))^2 d_u d_v},$$

where f is the corresponding eigenfunction, and $\sum_{\{u,v\}}$ denotes a sum over unordered pairs of vertices. Rewrite this as

$$\lambda_1 = \frac{2}{\mathrm{Vol}\ G}\frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_{(u,v)}(f(u)-f(v))^2 \frac{d_u d_v}{(\mathrm{Vol}\ G)^2}},$$

and observe that the sum in the denominator equals $\mathbb{E}[(f(u)-f(v))^2]$, where u, v are the random vertices as above. Thinking of $f(u)-f(v)$ as a telescoping sum over the edges of P, we use the identity $(\sum_{i=1}^k a_i)^2 \leq k\sum_{i=1}^k a_i^2$ to obtain

$$(f(u)-f(v))^2 \leq D\sum_{x\sim y}(f(x)-f(y))^2 1(\{x,y\} \in P).$$

Take the expectation of both sides gives and use (4):

$$\mathbb{E}[(f(u)-f(v))^2] \leq \frac{2D^2}{\mathrm{Vol}\ G}\sum_{x\sim y}(f(x)-f(y))^2.$$

This conveniently cancels the numerator of λ_1, and we are left with $\lambda_1 \geq \frac{1}{D^2}$ as desired. □