PUTNAM COMPETITION 1995A

A1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

A2. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

A3. The number $d_1d_2\dots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\dots e_9$ is such that each of the nine 9-digits numbers formed by replacing just one of the digits d_i in $d_1d_2\dots d_9$ by the corresponding digit e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1f_2\dots f_9$ is related to $e_1e_2\dots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7.

[For example, if $d_1d_2\dots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

A4. Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1,\ x_2,\ \dots\ ,\ x_n$ satisfy

$$\sum_{i=1}^{k} x_i \le k-1 \qquad \text{for } k = 1, 2, \dots, n.$$

A5. let $x_1, x_2, \dots, , x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned}
\frac{dx_1}{xt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,\\
\frac{dx_2}{xt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,\\
\vdots \qquad & \qquad\quad \vdots \qquad\qquad\qquad \vdots\\
\frac{dx_n}{xt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,
\end{aligned}$$

for some constants $a_{ij} \ge 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \dots, x_n$ necessarily linearly dependent?

A6. Suppose that each of n people writes down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \le b \le c$. Show that for some $n \ge 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.