PUTNAM COMPETITION 1994B

B1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. a is within n of b if $b - n \leq a \leq b + n$.)

B2. For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

B3. Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

B4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\lim_{n\to\infty} d_n = \infty$.

B5. For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

($\lfloor x \rfloor$ denots the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the k-fold composition of f_α.)

B6. For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100} \text{ implies } \{a, b\} = \{c, d\}.$$