PUTNAM COMPETITION 1994A

A1. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A2. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and ellipse $\frac{1}{9}x^2 + y^2 = 1$.

A3. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

A4. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

A5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}, \text{ with } i_1 < i_2 < \cdots < i_{1994}.$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

A6. Let $f_1, f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetations) which maps 0 to n. Consider the set of 1024 functions

$$\mathcal{F} + \left\{ f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} \;\middle|\; e_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10 \right\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathcal{F}$ map A to itself.