PUTNAM COMPETITION 1993B

B1. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

B2. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins?

B3. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

B4. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1,\ 0 \leq y \leq 1$, and the functions $f(x)$ and $G(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all $x,\ 0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y)dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

B5. Show that there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

B6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$, and replace them with $2x$ and $y - x$.