

On Multicolor Ramsey Numbers for Complete Bipartite Graphs

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INTRODUCTION

It follows from a well-known theorem of Ramsey [7] that for any finite graph G and any positive integer k , there exists a least integer $r(G; k)$ which has the following property.

Any k -coloring of the edges of the complete graph K_r on r edges always has a monochromatic subgraph isomorphic to G , provided only that $r \geq r(G; k)$.

Most work up to now has dealt with the case $k = 2$. The reader is invited to read Burr [2] for an excellent survey of the current state of affairs on this subject. In this paper, we examine the case in which G is the complete bipartite graph $K_{s,t}$ and k is arbitrary.

Without loss of generality we may assume $s \leq t$. For $s = 1$, the numbers $r(K_{1,t})$ are known exactly. They are given [3] by

$$r(K_{1,t}; k) = \begin{cases} k(t-1) + 1 & \text{if } k \equiv t \equiv 0 \pmod{2}, \\ k(t-1) + 2 & \text{otherwise.} \end{cases}$$

SOME UPPER BOUNDS

THEOREM 1.

$$r(K_{s,t}; k) \leq (t-1)(k + k^{1/s})^s \quad \text{for } k > 1, \quad t \geq s \geq 2.$$

Proof. We first obtain an upper bound on the number of edges e a graph G on n vertices may have if G contains no subgraph isomorphic to $K_{s,t}$. Let $M = (m_{ij})$ denote the adjacency matrix of G . Since $K_{s,t} \not\subseteq G$ then

$$\sum_{j=1}^n m_{i_1,j} \cdot m_{i_2,j} \cdots m_{i_s,j} \leq t - 1 \quad (1)$$

where $1 \leq i_1 < \cdots < i_s \leq n$. If c_j denotes $\sum_{i=1}^s m_{ij}$ then summing (1) over all choices of $i_1, \dots, i_s$, we obtain

$$\sum_{j=1}^n c_j(c_j - 1) \cdots (c_j - s + 1) \leq (t - 1) n(n - 1) \cdots (n - s + 1). \quad (2)$$

Since $f(x) = x(x - 1) \cdots (x - s + 1)$ is convex for $x > s - 1$, then (2) implies

$$nf\left(\frac{1}{n} \sum_{j=1}^n c_j\right) \leq (t - 1)f(n),$$

provided the argument of f exceeds $s - 1$. Since $e = \frac{1}{2} \sum_{j=1}^n c_j$, we have in this case

$$n((2e/n) - (s - 1))^s \leq (t - 1)n^s. \quad (3)$$

Now, for an arbitrary fixed $n \geq (t - 1)(k + k^{1/s})^s$, let the edges of K_n be k -colored. Thus, some color occurs on at least $(1/k)\binom{n}{2}$ edges. Let G denote the subgraph which has these edges. Since

$$k^{1/s}(k + k^{1/s})^{s-1} \geq k + 1$$

then

$$(t - 1)(k + k^{1/s})^s(1 - k/(k + k^{1/s})) > k(s - 1) + 1$$

for $t \geq s \geq 2$. But because of the assumption on n , we have

$$n(1 - k((t - 1)/n)^{1/s}) > k(s - 1) + 1$$

i.e.,

$$n - 1 > k(s - 1 + n((t - 1)/n)^{1/s}).$$

Thus, G has more than

$$\frac{1}{k} \binom{n}{2} > \frac{n}{2} \left(s - 1 + n \left(\frac{t - 1}{n} \right)^{1/s} \right)$$

edges. However, (3) can be rewritten as

$$e \leq (n/2)(s - 1 + n((t - 1)/n)^{1/s}) \quad (3')$$

and this is also clearly valid in the case that

$$\frac{1}{4} \sum_{j=1}^n c_j \leq s - 1.$$

Thus, by (3), G must contain a monochromatic copy of $K_{s,t}$. This proves the theorem. ■

A more careful argument can be used to prove the following somewhat stronger theorem.

THEOREM 1'.

$$r(K_{s,t}; k) \leq (t-1)k^s(1+e(k))^s \quad \text{for } k \geq 1, t \geq s \geq 2$$

where $e(k) = k^{1-s}(s-1+k^{-1})(t-1)^{-1}$.

For the special case $s = 2$, a closer analysis along the same lines can be used to establish the following result.

THEOREM 2.

$$r(K_{2,t}; k) \leq (t-1)k^2 + k + 2.$$

By a refinement of this argument for the case $t = 2$, one may obtain the following.

COROLLARY 1.

$$r(K_{2,2}; k) \leq k^2 + k + 1 \quad \text{for } k > 1.$$

As we shall see, this upper bound for the 4-cycle $K_{2,2}$ is fairly close to the known lower bound. The upper bound

$$r(K_{2,2}; k) < ck^2$$

for a suitable $c > 0$ had been previously obtained by Hajnal and Szemerédi (unpublished).

For the case $s = t$, Chvátal [6] has obtained the bound

$$r(K_{t,t}; k) \leq 2tk^t$$

which differs asymptotically from our bound for this case by a factor of 2.

SOME LOWER BOUNDS

We begin with a bound on $r(K_{2,2}; k)$.

THEOREM 3. For $k-1$ a prime power,

$$r(K_{2,2}; k) > k^2 - k + 1.$$

Proof. Since $k - 1$ is a prime power, then it is well known that there exists a simple difference set $D = \{d_1, \dots, d_k\}$ modulo $(k^2 - k + 1)$. For each t , $1 \leq t \leq k$, form a cyclic (symmetric) matrix $B_t = (b_t(i, j))$ as follows:

$$b_t(i, j) = \begin{cases} 1 & \text{if } i + j + d_t \equiv d_s \pmod{k^2 - k + 1} \text{ for some } d_s \in D, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Since D is a difference set, then it follows that for $i, j \in \mathbb{Z}_{k^2-k+1}$ (the integers modulo $(k^2 - k + 1)$), there exists a t such that $b_t(i, j) = 1$. Furthermore, for each t , no two rows of B_t have a common pair of 1's.

We now form a k -colored K_{k^2-k+1} as follows. The vertices of K_{k^2-k+1} will be the elements of $\mathbb{Z}_{k^2-k+1}$. The color of the edge $\{i, j\}$ for $i, j \in \mathbb{Z}_{k^2-k+1}$ is defined to be the least integer t such that $b_t(i, j) = 1$. By the preceding remark, no two rows of any B_t have a common pair of 1's and so, no monochromatic 4-cycle $K_{2,2}$ occurs in K_{k^2-k+1} . This shows that $r(K_{2,2}; k) > k^2 - k + 1$ and the theorem is proved. ■

A somewhat similar technique, based on n -dimensional projective geometries over finite fields, can be used to prove the following result:

$$r(K_{2,kn}; k) = k^{n+2} + o(k^{n+2}). \quad (5)$$

The details of the proof of (5) are a bit complicated and will not be given here (cf. [5]).

We remark that for two colors, it has been shown [4], [5] that

$$r(K_{2,t}; 2) \geq 4t - 2, \quad 4t - 3 \text{ a prime power.}$$

The best lower bound we know for the general case is given by a simple counting argument.

THEOREM 4.

$$r(K_{s,t}; k) > (2\pi \sqrt{st})^{1/(s+t)} ((s+t)/e^2) k^{(st-1)/(s+t)}.$$

Proof. Call a k -coloring of K_n *bad* if it contains a monochromatic $K_{s,t}$. It is easy to see that there are at most

$$\binom{n}{s+t} \binom{s+t}{s} k \cdot k^{\binom{n}{2}-st}$$

bad colorings. Hence, if this expression is less than the total number of k -colorings $k^{\binom{n}{2}}$ then we can deduce the inequality

$$r(K_{s,t}; k) > n.$$

Elementary calculations now show that if

$$n \leq (2\pi \sqrt{st})^{1/(s+t)} ((s+t)/e^2) k^{(st-1)/(s+t)}$$

where e denotes the base for natural logarithms, then

$$\binom{n}{s+t} \binom{s+t}{s} k^{\binom{n}{2}-st+1} < k^{\binom{n}{2}}$$

as required. This proves the theorem. ■

Note that for $t \gg s$, (6) becomes essentially

$$r(K_{s,t}; k) > (t/e^2) k^s \quad (6')$$

which is fairly close to the upper bound in Theorem 1.

CONCLUDING REMARKS

For a given graph G and integer $n \geq |G|$, define $T(G; n)$ to be the least integer m such that if H is any graph on n vertices with m edges then H must contain a subgraph isomorphic to G . These numbers are known as the *Turán* numbers for G . Clearly, if $R(G; n)$ denotes the minimum number of colors necessary to color K_n without forming a monochromatic G , then

$$R(G; n) > \binom{n}{2} / T(G; n). \quad (7)$$

Since

$$r(G; R(G; n) - 1) \leq n < r(G; R(G; n)) \quad (8)$$

then knowledge of $T(G; n)$ can be used to deduce bounds on $r(G; k)$. It was pointed out by Spencer [8] that in certain cases a simple probabilistic argument can be given which establishes *upper* bounds on $R(G; n)$. In particular, if $T(G; n) = o(n^2)$, then we have

$$R(G; n) = O((n^2 \log n) / T(G; n)). \quad (9)$$

For example, since it has been shown by Brown [1] that $T(K_{3,3}; n) = (n^{5/3}/2)(1 + o(1))$, then we can conclude

$$r(K_{3,3}; k) > ck^3 / \log^3 k \quad (10)$$

for some $c > 0$. Unfortunately, no very good bounds are currently known for $T(K_{r,s}; n)$.

It can also be shown using results from the theory of cyclotomy that

$$\lim_{t \rightarrow \infty} (1/t) r(K_{2,t}; k) = k^2.$$

The details may be found in [5].

It does not seem unreasonable to conjecture that in general, for $t \geq s \geq 2$,

$$r(K_{s,t}; k) \sim (t-1)k^s + o(k^s). \quad (11)$$

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