Math 262A/ CSE 291A
Midterm take home
Due: 2:30pm, Thursday, Nov. 2

1. What is the highest IQ in the world? More specifically, let $X_i, 1 \leq i \leq 5\times 10^9$ be independent, each normal with mean 100 and the standard deviation (square root of the variance) is 15. Let $X = \max_{1\leq i\leq 5\times 10^9} X_i$. Let $Y^{(\alpha)}$ denote the number of i with $X_i > \alpha$.

   (i) Find α with $E[Y^{(\alpha)}] \approx 1$
   (ii) Find β with $Pr[X < \beta] \approx .05$
   (iii) Find γ with $Pr[X < \gamma] \approx .95$

2. A set D of vertices in an Internet graph G is called dominating if every $v \in G$ is either in D or is adjacent to a vertex of D (or both). State (i.e. find the α in the statement) and prove a result of the following form.
   Theorem. If G has n vertices and all vertices have degree at least d then there exists a dominating set D with $|D| \leq \alpha$.

   To do this, let C be a random set of vertices with $Pr[v \in C] = p$. Let N be the set of vertices neither in C nor with any neighbors in C. Then $C \cup N$ is dominating. Adjust p so that $E[|C\cup N|]$ is small. Use the inequality $1-p \leq e^{-p}$ to simplify the analysis and give a cleaner statement.

3. Call a $0-1$ matrix A *ninefree* if there is no 3×3 submatrix with all entries one. (Rows and columns need not be consecutive.) Let $f(n)$ denote the maximal number of ones in an $n\times n$ ninefree A. Find a lower bound for $f(n)$ – i.e., show, for α as large as possible, that there exists a ninefree $n \times n$ A with at least α ones. Use the Deletion Method, first letting $Pr[a_{ij} = 1] = p$ and then changing a one to zero in every 3×3 submatrix with all entries one.

4. Let $A_1, \ldots , A_n \subset \{1, 2, \ldots , n\}$. We showed that (as a Army-Navy game) there exists $\chi : \{1, \ldots , n\} \to \{-1, 1\}$ so that all $|\chi(A_i)| \leq \sqrt{2n \log(2n)}$. Assume, for convenience, n even. Improve this result by using the following "pairing" distribution. Let $\chi(2i - 1)$ be uniform and independent, $1 \leq i \leq n/2$ and set $\chi(2i) = -\chi(2i - 1)$.