

Exercises from page 112 of the book by Alon and Spencer:

1. Let $G(V, E)$ be the graph whose vertices are all 7^n vectors of length n over Z_7 , in which two vertices are adjacent iff they differ in precisely one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G , and let W be the set of all vertices of G whose distance from U exceeds $(c + 2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n \cdot e^{-c^2/2}$.
2. Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V . Let $H = G[U]$ be the induced subgraph of G on U . Prove that

$$\Pr(\chi(H) \leq 400) < 1/100.$$

3. Prove that there is an absolute constant c such that for every $n > 1$ there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the probability that the chromatic number of $G(n, 1/2)$ lies in I_n is at least 0.99.