Math 261B

Notes Lecture 3 - 17 Jan 2012

Load Balancing

Suppose we have a scheduling problem with m jobs being fed to n processors. What is the maximum load if we assign jobs at random?

Case 1: $m = 6n\log n$. For $i \in [1,m]$ $P(X_i = 1) = \frac{1}{n}$. Define $X = \sum_{i=1}^{m} X_i$. $E[X_i] = \frac{1}{n}$ and $E[X] = \frac{m}{n} = 6\log n$. We want to determine the max load for a fixed processor.

Claim: max load $\leq 3E(x) = 18\log n$ with probability $1-\frac{1}{n}$. It suffices to show that $P(X > 18\log n) \leq \frac{1}{n^2}$.

TOOL: Let $X_1, ..., X_k$ be independent random indicator variables with $P(X_i = 1) = p_i$ for $1 \leq i \leq k$. Let $X = \sum_{i=1}^{k} X_i$ so $E(X) = \sum_{i=1}^{k} p_i$. Then for $\delta > 0$,

$$P(x > (1+\delta)E(X)) < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{E(X)}.$$

Note: δ is often taken to be 1, so this is not as scary as it looks.

Proof. For $t > 0$,

$$P(X > (1+\delta)E(X)) = P\left(e^{tX} > e^{(1+\delta)tE(X)}\right) \leq \frac{E(e^{tx})}{e^{(1+\delta)tE(X)}}$$

by Markov's Inequality.

$$E(e^{tX}) = E(e^{t\sum_i X_i}) = \prod_i E(e^{tX_i}) = \prod_i (p_i e^t + 1 - p_i) = \prod_i (1 + p_i(e^t - 1)) \leq \prod_i e^{p_i(e^t-1)}$$

using the fact that $1 + x \leq e^x$ for all $x \geq 0$. Continuing,

$$\prod_i e^{p_i(e^t-1)} = e^{\sum_i p_i(e^t-1)} = e^{(e^t-1)E(X)}.$$

Then choose t so that $e^t = 1+\delta$, and we have

$$\frac{E(e^{tx})}{e^{(1+\delta)tE(X)}} = \left(\frac{e^{(e^t-1)}}{e^{t(1+\delta)}}\right)^{E(X)} = \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{E(X)}.$$

□

Now use our tool with $\delta = 1$ to obtain

$$P(X > 3E(X)) \leq \left(\frac{e^2}{27}\right)^{6\log n} \leq \frac{1}{n^2}.$$

Case 2: $m = n$. Fix a processor, $X_1, ..., X_n$, $X = \sum_{i=1}^{n} X_i$, $E(X) = 1$. Then use the tool again,

$$P(X \geq (1+\delta)E(X)) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{E(X)}.$$

Let $c = 1+\delta$, so

$$P(X \geq c \cdot 1) \leq \left(\frac{e^{c-1}}{c^c}\right)^1 \leq \left(\frac{e}{c}\right)^c.$$

Our goal is to show that max load $= O\left(\frac{\log n}{\log\log n}\right)$. So we need to choose c so that $\left(\frac{e}{c}\right)^c < \frac{1}{n^2}$. Set $\gamma = \frac{c}{e}$ so we want $\gamma^{-e\gamma} = \frac{1}{n^2}$ or $\gamma^\gamma = n^{2/e}$. Take logs: $\gamma\log\gamma = \frac{2}{e}\log n$. Take logs again: $\log\gamma + \log\log\gamma = \log(2/e) + \log\log n$ so $2\log\gamma \geq \log\gamma + \log\log\gamma = \log(2/e) + \log\log n \approx \log\log n$. Going back to $\gamma\log\gamma = \frac{2}{e}\log n$ and divide through by $\log\log n$ and obtain $\frac{1}{2}\gamma \leq \frac{\gamma\log\gamma}{\log\log n} = \frac{2}{e}\frac{\log n}{\log\log n}$ so $\gamma = O\left(\frac{\log n}{\log\log n}\right)$.