Math 261B

Notes Lecture 12 - 16 Feb 2012

Martingales

We will introduce *martingales* through a series of examples.

Example 1: Flip a coin n times. Win \$1 with each heads and lose \$1 with tails . This gives random variables $X_i \in \{-1, 1\}, i = 1, ...n$. Let $S_j = \sum_{i \le j} X_i$, $S_0 = 0$. Then

$$E[S_n \mid S_1, S_2, ..., S_{n-1}] = E[S_n \mid S_{n-1}] = E[X_n + S_{n-1} \mid S_{n-1}] = E[X_n] + S_{n-1} = S_{n-1}.$$

This is our "definition" for a martingale.

Another key property is the Lipschitz condition, an analog for "continuity" in our discrete setting. We want to know that our random variables can't change too much in a single step. In this example,

$$|S_i - S_{i-1}| = |X_i| \le .1$$

The following is the main concentration inequality for martingales.

Theorem. *(Azuma-Hoeffding) Let $X_0, X_1, ...X_n$ be a martingale with Lipschitz condition $a_1 \le X_i - X_{i-1} \le a_i + c_i$. Then*

$$P(X_n > E[X_0] + \lambda) \le e^{\frac{-2\lambda^2}{\sum_{i=1}^n c_i^2}}.$$

Sometimes people like to combine two scary word, "sigma" and "algebra," into an absolutely terrifying word, "sigma-algebra." The main idea for us is that after each event (e.g. result of a coin flip) the size of the probability space is reduced, since there are fewer total possibilities as a result. This is clear in the setting of Example 1.

Similarly we can write

$$\begin{aligned} S_n &= X_1 + \cdots + X_{n-1} + X_n \\ S_{n-1} &= X_1 + X_2 + \cdots + X_{n-1} + E[X_n] \\ S_{n-2} &= X_1 + X_2 + \cdots + E[X_{n-1}] + E[X_n] \\ &\vdots \\ S_1 &= X_1 + E[X_2] + \cdots + E[X_{n-1}] + E[X_n] \\ S_0 &= E[X_1] + E[X_2] + \cdots + E[X_{n-1}] + E[X_n] \end{aligned}$$

This is another way of thinking about martingales.

Example 2: (Balls in Bins) We have m balls and n bins. We want to know the number of empty bins. Let $Z_i = \begin{cases} 1 & \text{if bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$, $Z = \sum_{i=1}^n Z_i$. Then

$$\mu = E(Z) = nP(\text{a bin is empty}) = n\left(1 - \frac{1}{n}\right)^m$$

so $\mu \approx ne^{-m/n}$ when m is large. Then using Azuma-Hoeffding with $c_i = 1 \forall i$,

$$P(|Z - \mu| > t) \le 2e^{-\frac{2t^2}{m}}.$$