Math 261B

Notes Lectures 1 & 2

Amplification

Suppose $X_1, X_2, ...$ are indicator random variables, and assume they are independent. Suppose further that $P(\text{success}) = P(X_i = 1) = p = \frac{3}{4}$. By repeating the event several times, we can take this probability of success that is just over $\frac{1}{2}$ and obtain confidence nearing 100%. Let $X = X_1 + X_2 + \cdots + X_n$. Then

$$E(X) = \sum_{i=1}^{n} E(X_i) = \frac{3}{4}n$$

by Linearity of Expectation. Then

$$P\left(X < \frac{n}{2}\right) \leq e^{-n/8},$$

i.e., the probability of failure is very small for large n.

We will prove two concentration inequalities, the second of which we will use to prove the above result:

$$P(X > E(X) + \delta n) \leq e^{-2\delta^2 n}$$

and

$$P(X < E(X) - \delta n) \leq e^{-2\delta^2 n}.$$

Proof. For any $t > 0$,

$$P(X > E(X) + \delta n) = P(e^{tX} > e^{tE(X) + t\delta n}) \leq \frac{E(e^{tX})}{e^{tE(X)+t\delta n}} = \frac{E(e^{tX})}{e^{tn(p+\delta)}}$$

by Markov's Inequality. Also, by independence of the X_i's,

$$E\left(e^{tX}\right) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = \prod_{i=1}^{n} E\left(e^{tX_i}\right) = (pe^t + q)^n \quad \text{where } q = 1 - p.$$

So

$$P(X > E(X) + \delta n) \leq \left(\frac{pe^t + q}{e^{t(p+\delta)}}\right)^n.$$

We wish to choose t to minimize $f(t) = \dfrac{pe^t + q}{e^{t(p+\delta)}}$. To do this we set

$$f'(t) = \frac{pe^t}{e^{t(p+\delta)}} - \frac{(pe^t + q)(p+\delta)}{e^{t(p+\delta)}} = 0$$

which simplifies to $pe^t(1 - p - \delta) = q(p + \delta)$ so the minimum occurs when $e^t = \dfrac{q(p+\delta)}{p(q-\delta)}$. We plug this in to $f(t)$ to obtain the minimum

$$\frac{\frac{q(p+\delta)}{q-\delta} + q}{\left(\frac{q}{p}\frac{p+\delta}{q-\delta}\right)^{p+\delta}} = \left(\left(\frac{q}{q-\delta}\right)^{q-\delta}\left(\frac{p}{p+\delta}\right)^{p+\delta}\right)^n.$$

We want to know if this is less than or equal to $e^{-2\delta^2 n}$. In other words, taking logs, we want to see if

$$g(\delta) = (q-\delta)\log\frac{q}{q-\delta} + (p+q)\log\frac{p}{p+\delta} + 2\delta^2 \leq 0?$$

Notice that $g(0) = 0$ so we just need to show $g'(\delta) < 0$. Note that $g'(\delta) = \log\dfrac{p}{p+\delta} - 1 - \log\dfrac{q}{q-\delta} + 1 + 4\delta$. So $g'(0) = 0$ and $g''(\delta) = -\frac{1}{p+\delta} - \frac{1}{q-\delta} + 4$, so $g''(\delta) \leq 0$ using the fact that if $a + b = 1$ then $\dfrac{1}{a} + \dfrac{1}{b} \geq 4$.

Therefore using the intermediate theorem, we write $g(\delta) = g(0) + g'(0)\delta + g''(\xi)\delta^2 \leq 0$ and the first inequality follows.

To prove the second inequality, $P(X < E(X) - \delta n) \leq e^{-2\delta^2 n}$, we proceed in a similar manner: Set $Y_i = -X_i$ and $Y = \sum_{i=1}^n Y_i$. Then $P(X < E(X) - \delta n) = P(Y > E(Y) + \delta n)$. Then for any $t > 0$,

$$P(Y > E(Y) + \delta n) = P(e^{tY} > e^{tE(Y)+t\delta n}) \leq \frac{E(e^{tY})}{e^{tn(-p+\delta)}} = \frac{\prod E(e^{tY_i})}{e^{tn(-p+\delta)}} = \left(\frac{pe^{-t} + q}{e^{t(-p+\delta)}}\right)^n.$$

So we wish to minimize $f(t) = \dfrac{pe^{-t} + q}{e^{t(-p+q)}}$. Set

$$f'(t) = \frac{-pe^{-t}}{e^{t(-p+\delta)}} + \frac{(p-\delta)(pe^{-t} + q)}{e^{t(-p+\delta)}} = 0.$$

Solving this, the minimum occurs when $e^{-t} = \dfrac{q(p-\delta)}{p(q+\delta)}$. (We should also check $f''(t)$ to ensure that this is a minimum, but we will leave out this detail.) Plugging this into $f(t)$ yields a minimum value of $\left(\left(\dfrac{q}{q+\delta}\right)^{q+\delta}\left(\dfrac{p-\delta}{p}\right)^{-p+\delta}\right)^n$. We want to verify that this is less than or equal to $e^{-2\delta^2 n}$. As above, taking logs, this amounts to showing

$$g(\delta) = (q+\delta)\log\frac{q}{q+\delta} + (-p+\delta)\log\frac{p-\delta}{p} - 2\delta^2 \leq 0.$$

Notice $g(0) = 0$ and $g'(\delta) = \log\dfrac{q}{q+\delta} + \log\dfrac{p-\delta}{p} + 4\delta$. Then $g'(0) = 0$ and $g''(\delta) = -\frac{1}{q+\delta} - \frac{1}{p-\delta} + 4 \leq 0$ as above. So $g'(0) = 0$ and $g''(\delta) \leq 0 \Rightarrow g'(\delta) \leq 0$, and $g(0) = 0$ and $g'(\delta) \leq 0 \Rightarrow g(\delta) \leq 0$, and the result follows. □

Using this second inequality in our example, with $\delta = \frac{1}{4}$ we see that

$$P\left(X < \frac{n}{2}\right) \leq e^{-n/8}$$

as claimed above. Hence, with repeated trials, the probability of failing more than half the time goes to zero very quickly. If we take probability of success $p = \frac{1}{2} + \epsilon$ for any $\epsilon > 0$, then similar reasoning gives that the probability of failure more than half the time goes to zero exponentially as n goes to infinity.