Lecture 9: Lovász Local Lemma

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Theorem 1 (Asymmetric Lovász Local Lemma). *Let $A_1, ..., A_n$ be events in some probability space. Let $D = (V, E)$ be the dependency digraph where A_i is independent of $\{A_j : (i,j) \notin E\}$. If $0 \leq x_i < 1$ so that $P(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ then $P(\bigcap_{i=1}^n \neg A_i) \geq \prod_{i=1}^n (1 - x_i) > 0$.*

Proof. Suppose we can show $P(A_i | \bigcap_{j\in S} \neg A_j) \leq x_i$ where $S \subset [n] - \{j\}$. Then

$$P(\bigcap_{i=1}^{n} \neg A_i) = (1 - P(A_1))(1 - P(A_2|\neg A_1)...(1 - P(A_n| \bigcap_{j=1}^{n-1} \neg A_j))$$

So we need to prove

$$P(A_i| \bigcap_{j\in S} \neg A_j) \leq x_i, \text{ where } S \subset \{A_1, A_2, ..., A_n\} \tag{1}$$

We can do so by induction on the size of S. For $|S| = 0$,

$$P(A_i) \leq x_i \prod_{(i,j)\in E} (1 - x_j) < x_i$$

Now assume (1) holds for sets up to size $k - 1$. We break S into two sets:

$$S_1 = \{j \in S | (i,j) \in E\}$$

$$S_2 = S \backslash S_1$$

Assume that i is in S_1 because otherwise we are reduced to the base case. Then

$$P(A_i|(\bigcap_{j\in S_1} \neg A_j) \cap (\bigcap_{k\in S_2} \neg A_k))) = \frac{P(A_i \cap (\bigcap_{j\in S_1} \neg A_j) \cap (\bigcap_{k\in S_2} \neg A_k)))P(\bigcap_{k\in S_2} \neg A_k)}{P((\bigcap_{j\in S_1} \neg A_j) \cap (\bigcap_{k\in S_2} \neg A_k))P(\bigcap_{k\in S_2} \neg A_k)}$$

$$= \frac{P(A_i \cap (\bigcap_{j\in S_1} \neg A_j)|(\bigcap_{k\in S_2} \neg A_k)))}{P((\bigcap_{j\in S_1} \neg A_j)|(\bigcap_{k\in S_2} \neg A_k))} \tag{2}$$

First we bound the numerator of (2):

$$P(A_i \cap (\bigcap_{j \in S_1} \neg A_j) | (\bigcap_{k \in S_2} \neg A_k))) \leq P(A_i | \bigcap_{k \in S_2} \neg A_k) = P(A_i)$$

$$\leq x_i \prod_{(i,j) \in E} (1 - x_j)$$

where the first inequality follows because A_i is independent of the events in S_2.

Next we bound the numerator in the following way. Let $S_1 = \{A_{j_1}, A_{j_2}, ..., A_{j_{|S_1|}}\}$. Then

$$P(\bigcap_{j \in S_1} \neg A_j | \bigcap_{k \in S_2} \neg A_k) = P(\neg A_{j_1} | \bigcap_{k \in S_2} \neg A_k) \cdot P(\neg A_{j_2} | \neg A_{j_1} \cap \bigcap_{k \in S_2} \neg A_k) \dots$$

$$\geq (1 - x_{j_1})(1 - x_{j_2}) \dots (1 - x_{j_{|S_1|}})$$

The inequality follows from the inductive hypothesis because there are at most $k-1$ events on the right hand side of each conditional probability.

Now notice that each term in the denominator also appears in the bound on the numerator. Thus, combining the two bounds completes the proof of (1). □

Theorem 2 (Symmetric Lovász Local Lemma)**.** *Let $A_1, ..., A_n$ be events in a probability space, suppose each A_i is mutually independent of at least $n - (d+1)$ other events, and $P(A_i) \leq p$. If $ep(d+1) \leq 1$, then $P(\bigcap_{i=1}^{n} \neg A_i) > 0$.*

Proof. The symmetric version of the local lemma follows immediately from the asymmetric version in the following way. Let $x_i = \frac{1}{d+1}$.

$$p \leq \frac{1}{d+1} \prod_{(i,j) \in E} (1 - \frac{1}{d+1}) \leq \frac{1}{d+1} (1 - \frac{1}{d+1})^d \leq \frac{1}{e(d+1)}$$ □

0.1 Application to Ramsey Number

Definition 1. *The Ramsey number $R(s,t)$ is the least n, such that any red, blue coloring of the edges of K_n contains either a red K_s or a blue K_t.*

Consider $R(t,3)$. Color each edge blue with probability p. A_T for $|T| = 3$, is the event that T is blue. B_S for $|S| = t$, is the event that S is red.

Using the Lovász Local Lemma, we get that $R(t,3) > \frac{c_3 t^2}{\log t}$

0.2 Algorithmic Lovász Local Lemma

Let Ω be a set of size n and X_v for $r \in \Omega$ be independent random variables. Let $e^1, ..., e^m \subset \Omega$ and $B_1, ..., B_m$ be events so that B_i depends only on X_v for $v \in e^i$. We say that e^i and e^j overlap if $e^i \bigcap e^j \neq \emptyset$.

Theorem 3. *If each event overlaps at most d others, $P(B_j) \leq p\ \forall j$ and $p\frac{d^d}{(d-1)^{d-1}} \leq 1$, then there is a randomized algorithm which finds an assignment to $\{X_v\}$ so that none of the B_i occur and it runs in expected linear time.*

Algorithm:

Give X_v these values randomly.

While some event B_i holds:

1. Pick one such (k).
2. Redo the X_v for e^k.

Proof idea: the expected number of times you see e^i is $\frac{1}{d-1}$. There are m events, so the expected total time is $\frac{m}{d-1} \leq \frac{dn}{d-1}$.

The full proof is in the next lecture.