*Comp. & Maths. with Appls.* Vol. 1, pp. 365–369. Pergamon Press, 1975. Printed in Great Britain.

# ON SPARSE GRAPHS WITH DENSE LONG PATHS

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(*Received April* 1975)

## INTRODUCTION

The following problem was raised by H.-J. Stoss[3] in connection with certain questions related to the complexity of Boolean functions. An acyclic directed graph $G$ is said to have property $P(m, n)$ if for any set $X$ of $m$ vertices of $G$, there is a directed path of length $n$ in $G$ which does not intersect $X$. Let $f(m, n)$ denote the minimum number of edges a graph with property $P(m, n)$ can have. The problem is to estimate $f(m, n)$.

In this paper we shall restrict ourselves to the case $m = n$. We shall prove

$$c_1 n \log n/\log\log n < f(n, n) < c_2 n \log n \tag{1}$$

(where $c_1, c_2, \ldots$, will hereafter denote suitable positive constants). In fact, the graph we construct in order to establish the upper bound on $f(n, n)$ in (1) will have just $c_3 n$ vertices. In this case the upper bound in (1) is essentially best possible since it will also be shown that for $c_4$ sufficiently large, if a graph on $c_4 n$ vertices has property $P(n, n)$ then it must have at least $c_5 n \log n$ edges.

## A PRELIMINARY LEMMA

In order to establish the upper bound in (1) we first need the following result.

*Lemma.* For all $\delta > 0$ there exists $c = c(\delta)$ such that for all $t$ sufficiently large, there exists a bipartite graph $B = B(\delta; t)$ with vertex sets $A$ and $A'$ so that:

(i) $|A| = |A'| = t$;

(ii) $B$ has at most $c(\delta)t$ edges;

(iii) If $X \subseteq A$, $X' \subseteq A'$ with $|X| \geq \delta t$, $|X'| \geq \delta t$ then $(X, X') = \{\{x, x'\} : x \in X, x' \in X'\}$ contains an edge of $B$.

*Proof*: We use a simple probabilistic argument to show the existence of $B$. Form a bipartite graph $\bar{B}$ on the vertex sets $A$ and $A'$ with $|A| = |A'| = t$ by selecting for each $a \in A$ a *random* subset $\bar{B}(a) \subseteq A'$ of cardinality $d = d(\delta)$ (to be specified later). Call $\bar{B}$ "bad" if there exists $X \subseteq A$, $X' \subseteq A'$, with $|X| \geq \delta t$, $|X'| \geq \delta t$, so that $(X, X')$ contains no edge of $\bar{B}$. For *fixed* $X$ and $X'$, the probability that $\bar{B}$ is bad because of these two subsets is at most

$$\binom{(1-\delta)t}{d}^{\delta t}\binom{t}{d}^{(1-\delta)t}\Big/\binom{t}{d}^{t}.$$

Hence, the total probability that $\bar{B}$ is bad is at most

$$\binom{t}{\delta t}^{2}\binom{(1-\delta)t}{d}^{\delta t}\binom{t}{d}^{(1-\delta)t}\Big/\binom{t}{d}^{t}.$$

A simple computation shows that if $d$ is chosen suitably large, for example, so that

$$(1-\delta^2)^{d\delta} < 1/4,$$

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This research was supported in part by National Science Foundation grant GJ 36473X, by the Office of Naval Research contract NR 044-402, and by IBM Corporation. Reproduction in whole or in part is permitted for any purpose of the United States Government.

then for $t$ sufficiently large (e.g., $t > d/\delta$) this probability is less than 1, and so, a graph $B = B(\delta; t)$ must exist which satisfies the requirements of the lemma.

## CONSTRUCTION OF G

The next step in the proof of (1) is the construction of the directed graph $G$. For large $n$, $G = G(n)$ will have as its vertex set $V = \{0, 1, \ldots, 2^n - 1\}$. If $v$ and $m$ are positive integers, then $D_v(m)$ will denote the set $\{v, v+1, \ldots, v+m-1\} \cap V$. Similarly, $D_v^*(m)$ will denote the set $\{v, v-1, \ldots, v-m+1\} \cap V$. In general, $\epsilon_1, \epsilon_2, \ldots$, will denote suitably chosen fixed positive constants to be specified later. The edge set $E$ of $G$ is formed as follows:

(i) For $v \epsilon V$, the pairs $(v, x)$, $x \epsilon D_{v+1}(4n)$, are in $E$;

(ii) For each $t$ with $n/2 \leq 2^t < 2^n$, and each $i$ as specified below a copy of $B(\epsilon_1; 2^t)$ is formed between the vertex sets $A = D_{m.2^t}(2^t)$ and $A' = D_{(m+i).2^t}(2^t)$, $0 \leq m < 2^{n-t}$, where $i = 1, 2, \ldots, 10$ (or if $i$ cannot assume the value 10 because $(m+10)2^t > 2^n$, then it ranges from 1 to $2^{n-t} - m$). All edges are directed from $x$ to $y$ with $x < y$.

An elementary calculation shows that

$$|E| < c_6 n 2^n.$$

## THE UPPER BOUND

*Theorem* 1. For a suitable $\epsilon > 0$, $G(n)$ has property $P(\epsilon . 2^n, \epsilon . 2^n)$ for all sufficiently large $n$.

*Proof*: The theorem will be proved by a sequence of claims. First we show that $G(n)$ shares with the graphs $B(\epsilon; t)$ the following property.

*Claim* 1. If $m \geq 2n$ and $X \subseteq D_x(m)$, $X' \subseteq D_{x+m}(m)$, satisfy $|X| \geq \epsilon_2 m$, $|X'| \geq \epsilon_2 m$, then $[X, X'] = \{(x, x') : x \epsilon X, x' \epsilon X'\}$ contains an edge of $G(n)$.

*Proof of Claim*: Let $2^t \leq m/2 < 2^{t+1}$. Thus, $m/4 < 2^t$ so at most *five* of the intervals $D_{r.2^t}(2^t)$ intersect $D_x(m)$ and at most five of them intersect $D_{x+m}(m)$. Since $|X| \geq \epsilon_2 m$ then some $D_{r.2^t}(2^t)$ and $D_{r'.2^t}(2^t)$ have

$$|D_{r.2^t}(2^t) \cap X| \geq \epsilon_2 m/5, |D_{r'.2^t}(2^t) \cap X'| \geq \epsilon_2 m/5. \tag{3}$$

But we must have $|r' - r| \leq 10$ so that by the construction of $G(n)$ there is a copy of $B(\epsilon_1; 2^t)$ between $D_{r.2^t}(2^t)$ and $D_{r'.2^t}(2^t)$. Thus, if $\epsilon_2/5 > \epsilon_1$ and $m \geq 2^t$ then the property of $B(\epsilon_1; 2^t)$ guaranteed by the Lemma implies that $[X, X']$ contains an edge of $G(n)$ provided that $t$ is sufficiently large (which is guaranteed by choosing $n$ large enough). This proves the claim.

Next, let us choose an arbitrary fixed set $X$ of vertices with $|X| \leq \epsilon . 2^n$. The vertices in $X$ will be referred to as the *marked* vertices of $G$; the remaining vertices of $G$ will be called the *unmarked* vertices of $G$.

Let us call an unmarked vertex $y \epsilon V$ *bad* if for some $m \geq 1$ either at least $\epsilon_3 m$ vertices in $D_y(m)$ are marked or at least $\epsilon_3 m$ vertices in $D_y^*(m)$ are marked. Otherwise, an unmarked vertex of $G$ is called *good*.

*Claim* 2. There are at most $\epsilon_4 2^n$ bad vertices.

*Proof of Claim*: Let $y_1$ denote the least unmarked vertex of $G$ (if it exists) for which for some $m_1 \geq 1$, at least $\epsilon_3 m_1$ vertices in $D_{y_1}(m_1)$ are marked. In general, if $y_1, \ldots, y_k$ and $m_1, \ldots, m_k$ have been defined, let $y_{k+1}$ be the least unmarked vertex of $G$ following $y_k + m_k - 1$ (if it exists) for which for some $m_{k+1} \geq 1$ at least $\epsilon_3 m_{k+1}$ vertices in $D_{y_{k+1}}(m_{k+1})$ are marked. We continue this process until it no longer can be applied, so that, say, $y_1, \ldots, y_s$ and $m_1, \ldots, m_s$ have been defined. Similarly, let $y_1^*$ denote the greatest unmarked vertex (if it exists) for which for some $m_1^* \geq 1$, at least $\epsilon_3 m_1^*$ vertices in $D_{y_1^*}^*(m_1^*)$ are marked, etc. In this way, we define $y_1^*, \ldots, y_{s^*}^*$ and $m_1^*, \ldots, m_{s^*}^*$.

It follows from the preceding construction and the definition of a bad vertex that *all* bad vertices are contained in the set

$$Y = \bigcup_{k=1}^{s} D_{y_k}(m_k) \cup \bigcup_{k=1}^{s^*} D_{y_k^*}^*(m_k^*)$$

Thus, there are at most

$$M = \sum_{k=1}^{s} m_k + \sum_{k=1}^{s^*} m_k^*$$

bad vertices. However, by our construction there are at least $(\epsilon_3/2)M$ marked vertices in $Y$. Since by hypothesis there are at most $\epsilon . 2^n$ marked vertices in $V$ then we have

$$(\epsilon_3/2)M \leq \epsilon . 2^n,$$

$$M \leq (2\epsilon/\epsilon_3)2^n < \epsilon_4 2^n,$$

which proves the claim.

For an unmarked vertex $x$, let $P_x(m)$ denote the set of all unmarked vertices in $D_x(m)$ which can be reached from $x$ by directed paths which contain only unmarked vertices.

*Claim* 3. If $x$ is a good vertex and $|D_x(m)| = m$ then

$$|P_x(m)| > \epsilon_5 m \tag{4}$$

*Proof of Claim*: If $m \leq 4n$ then since $x$ is good, at least $(1-\epsilon_3)m$ vertices in $D_x(m)$ are unmarked and $x$ has edges directly to all of them. Suppose $m > 4n$. Let $m'$ denote $[m/2]$. Since $|D_x(m')| = m'$ then by induction $|P_x(m')| > \epsilon_5 m'$. Since $x$ is good then at most $\epsilon_3 m$ vertices in $D_x(m)$ are marked. Hence, at most $\epsilon_3 m$ vertices in $D_{x+m'}(m') \subseteq D_x(m)$ are marked. Since $m' \geq 2n$ and $\epsilon_5 \geq \epsilon_2$ then there are edges from $P_x(m')$ to at least $(1-\epsilon_2)m'$ vertices in $D_{x+m'}(m')$. But at most $\epsilon_3 m < 3\epsilon_3 m'$ vertices in $D_{x+m'}(m')$ are marked. Hence, $P_x(m')$ must have edges to at least $(1-\epsilon_2-3\epsilon_3)m'$ *unmarked* vertices in $D_{x+m'}(m')$. Since $1-\epsilon_2-3\epsilon_3 > 3\epsilon_5$ then

$$|P_x(m)| > 3\epsilon_5 m' > \epsilon_5 m.$$

The claim now follows by induction.

In exactly the same way if follows that if $P_x^*(m)$ denotes the set of all unmarked vertices in $D_x^*(m)$ which are connected to the unmarked vertex $x$ by a directed path containing only unmarked vertices, and $x$ is a good vertex and $D_x^*(m) = m$, then

$$|P_x^*(m)| > \epsilon_5 m. \tag{4'}$$

*Claim* 4. Let $x$ and $x'$ be good vertices with $x < x'$. Then $x' \epsilon P_x(2^n)$.

*Proof*: If $x' - x \leq 4n$ then the claim is immediate since by construction there is an edge from $x$ to $x'$. Assume $x' - x > 4n$. Let $y = [(x + x')/2]$ and let $m = y - x + 1$. Consider the intervals $D_x(m)$ and $D_{x'}^*(m)$. Either they are adjacent or they have the single element $y$ in common. Since $x$ and $x'$ are good then by (4) and (4')

$$|P_x(m)| > \epsilon_5 m, |P_{x'}^*(m)| > \epsilon_5 m. \tag{5}$$

Since $\epsilon_5 \geq \epsilon_2$ then by Claim 1, there is an edge in $G$ from a vertex of $P_x(m)$ to a vertex of $P_{x'}^*(m)$. Thus, there is a directed path from $x$ to $x'$ containing no marked vertices and the claim is proved.

The proof of the theorem is now immediate. By Claim 2 there are at least $(1-\epsilon_4-\epsilon)2^n$ good vertices in $G$. By Claim 4 we can form a directed path which contains only unmarked vertices and which contains *all* the good vertices (since $x'$ can always be chosen to be the next good vertex following $x$). Since $1-\epsilon_4-\epsilon > \epsilon$ then the theorem follows (where it is easily seen how the appropriate values of $\epsilon_k$ and $c_k$ can be chosen).

## THE LOWER BOUND

The following result will establish the lower bound in (1).

*Theorem* 2. Let $H$ be an acyclic directed graph with at most $c_7 n \log n / \log\log n$ edges where $n$ is a large fixed integer. Then there is a set of at most $n$ vertices of $H$ which hits every directed path of length $n$.

*Proof*: Let us denote the vertex set of $H$ by $V = \{1, 2, \ldots, v\}$. We may assume that all edges are of the form $(i, j)$ with $i < j$. For an edge $e = (i, j)$ of $H$, let *length* $(e)$ be defined to be $j - i$.

Partition the edges of $H$ into classes $C_0, C_1, \ldots, C_r$ where

$$C_k = \{e: 2^{4k \log\log n} \leq \text{length}(e) < 2^{4(k+1)\log\log n}\}$$

and $r = [\log v/4 \log\log n]$.

Since $H$ has at least $c_8 n \log n/\log\log n$ edges then it follows that $v \geq c_9 n^{1/2}$ and $r \geq c_{10} \log n/\log\log n$. Hence some class $C_a$ with $0 \leq a < r$ has at most $c_{11} n$ elements. Let us delete all vertices in $H$ incident to any of the edges in $C_a$. Furthermore, we also delete those vertices $x \in V$ which satisfy

$$0 \leq x - m \cdot 2^{4a \log\log n}(1 + 2^{2 \log\log n}) < 2^{4a \log\log n}$$

for some integer $m \geq 0$. This latter step removes at most

$$\left(\frac{2}{2^{2 \log\log n} - 1}\right) v = 0(n)$$

vertices, since $v \leq 2c_7 n \log n/\log\log n$. Hence we have deleted at most $c_{12} n$ vertices altogether. However, any directed path remaining has at most

$$\left(\frac{2^{(4a+2)\log\log n} - 2^{4a \log\log n}}{2^{4(a+1)\log\log n}}\right) v = 0(n)$$

edges, since we cannot go more than $2^{(4a+2)\log\log n} - 2^{4a \log\log n}$ steps without using an edge whose length exceeds $2^{4a \log\log n}$; and the length of such an edge actually exceeds $2^{4(a+1)\log\log n}$. This proves the theorem.

By using a different partition of the edges of $H$, namely, into the classes $C_0', \ldots, C_r'$ where

$$C_k' = \{e: 2^{c_{13}k} \leq \text{length}(e) < 2^{c_{13}(k+1)}\}$$

for a suitable constant $c_{13}$, we can establish the following result.

*Theorem* 3. If $c_{14}$ is sufficiently large then any graph $G$ on $c_{14} n$ vertices having property $P(n, n)$ must have at least $c_{15} n \log n$ edges.

The graphs $G(n)$ used in Theorem 1 show that the result in Theorem 3 is to within constant factors best possible.

## SOME RELATED QUESTIONS

We now consider several problems for ordinary (undirected) graphs. Let $F_e(n, n)$ (resp., $F_v(n, n)$) denote the smallest integer for which there is a graph with $F_e(n, n)$ edges so that the deletion of any $n$ of its vertices there still remains a connected component of $n$ edges (resp., vertices). We shall prove by probabilistic methods that

$$F_e(n, n) < c_{16} n, F_v(n, n) < c_{17} n. \tag{6}$$

The method we use is the same as that in the work of Erdös and Renyi[1], [2]. It turns out that almost all graphs have the desired property.

*Theorem* 4. For every $\epsilon > 0$ there is a $c = c(\epsilon)$ so that all but $0\left(\binom{\binom{(2+\epsilon)n}{2}}{cn}\right)$ graphs $G$ with $(2+\epsilon)n$ vertices and $cn$ edges have the property that after the omission of any $n$ of its vertices, a connected component of at least $n$ vertices remains.

*Proof*: It suffices to show that if $n$ vertices are omitted and the remaining $n(1+\epsilon)$ vertices are split into two classes $S_1$ and $S_2$ with $|S_1| \geq \epsilon n$, $|S_2| \geq \epsilon n$, then there is at least one edge joining a vertex of $S_1$ to a vertex of $S_2$.

Consider a random graph $G$ on $(2+\epsilon)n$ vertices and $cn$ edges (where $c$ will be specified later). There are $\binom{(2+\epsilon)n}{n}$ ways that $n$ vertices of $G$ can be deleted. The remaining $n(1+\epsilon)$ points

can then be split into two sets $S_1$ and $S_2$ in at most $2^{n(1+\epsilon)}$ ways. Thus, the total number of splittings is at most

$$\binom{(2+\epsilon)n}{n}2^{n(1+\epsilon)} < 2^{(2+\epsilon)n}2^{n(1+\epsilon)} < 2^{3(1+\epsilon)n}$$

Between $S_1$ and $S_2$ there are at least $\epsilon n^2$ potential edges. The probability that none of these edges actually occurs in $G$ is less than $\left(1-\frac{c}{(2+\epsilon)n}\right)^{\epsilon n^2}$. Thus, if $c$ is chosen so that

$$2^{3(1+\epsilon)n}\left(1-\frac{c}{(2+\epsilon)n}\right)^{\epsilon n^2} \to 0 \tag{7}$$

as $n \to \infty$ then we easily see that almost all graphs cannot be split in such a way.

Since

$$\left(1-\frac{c}{(2+\epsilon)n}\right)^{\epsilon n^2} \to e^{-(\epsilon c/(2+\epsilon))n}$$

then for $c$ large enough, e.g., $c > 18(\epsilon + \epsilon^{-1})$,

$$e^{-(\epsilon c/(2+\epsilon))n} < e^{-3(1+\epsilon)n}$$

and (7) holds. This proves the theorem.

The other half of (6) is proved in a similar way. It would be interesting to determine the best possible value of $c$ but this does seem to be too easy.

We mention here the undirected analogue of (1). Let $g(n, n)$ denote the smallest integer for which there is an undirected graph of $g(n, n)$ edges so that if we omit any $n$ of its vertices then there always remains a path of length $n$. We believe

$$\frac{g(n,n)}{n} \to \infty, \qquad \frac{g(n,n)}{n \log n} \to 0$$

as $n \to \infty$ and hope to return to this question in finite time.

A related question is the following: Consider random graphs on $n$ vertices and $Cn$ edges. Is it true that for large $C$ almost all of these graphs have a path of length $n(1-\epsilon)$? It is known [4] that almost all graphs on $n$ vertices and $(1/2+\epsilon)\, n \log n$ edges are Hamiltonian.

It is possible to introduce another parameter into these questions. Let $F_v(t; n, n)$ denote the smallest integer for which there is a graph with $t$ vertices and $F_e(t; n, n)$ edges having the property that if any $n$ vertices are deleted there still remains a connected component with at least $n$ vertices. If $t/n \to c > 2$ then $F_v(t; n, n)/n \to A(c)$ where $A(c) \to \infty$ as $c \to 2$. (The behavior of $F_e(t; n, n)/n$ is similar). We would also omit edges instead of vertices but leave the formulation of these questions to the reader.

*Acknowledgment*—The authors gratefully acknowledge the useful suggestions on this problems given to us by D. E. Knuth.

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