Open problems in Euclidean Ramsey Theory

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Abstract

Euclidean Ramsey theory typically deals with geometrical problems in which some unavoidable structure remains whenever a sufficiently large (geometrical) object is partitioned into finitely many parts. In this note, we survey a number of open problems in this subject.

1 Introduction

Ramsey theory is a branch of combinatorics that has often been characterized by the phrase "Complete disorder is impossible". More precisely, it is the study of structure which must be retained no matter how a sufficiently large object is partitioned into a (usually) finite number of parts [5]. Classic theorems of this type are Ramsey's theorem, van der Waerden's theorem on arithmetic progressions, Schur's theorem (on integer solutions of $x + y = z$,

*Research supported in part by NSF Grant No. CCR 031-0991

the Hales-Jewett theorem (on combinatorial lines), for example, as well as the (pre-historic) Pigeon-hole principle. In Euclidean Ramsey theory, the structures and properties are geometrical in nature. For example, it is true that if the points in Euclidean 5-space are partitioned into two sets then one of the sets must contain the vertices of a unit square. More generally, for any integer r, if the points of Euclidean n-space are partitioned into r parts, and n is sufficiently large, then at least one of the parts must contain the vertices of a unit square. Just how the minimum value of n for which this is true depends on r is still not known. This is a typical problem in Euclidean Ramsey theory, and we will discuss this and many others in subsequent sections.

We first give some basic definitions. For a finite set $X \subset \mathbb{E}^k$, let *Cong(X)* denote the set of all subsets of $\mathbb{E}^k$ which are *congruent* to X under some Euclidean motion. We will say that X is *Ramsey* if for any integer r, there is a least integer $N(X, r)$ such that if $N \geqslant N(X, r)$ then for any partition of $\mathbb{E}^N = C_1 \cup C_2 \cup \ldots \cup C_r$, we have $X' \subset C_i$ for some $X' \in Cong(X)$ and some i. If we think of the partition as an r-coloring of $\mathbb{E}^N$, then we often say that $\mathbb{E}^N$ contains a *monochromatic* copy of X. We denote this property by the usual "arrow" notation $\mathbb{E}^N \rightarrow X$. The negation of this statement is denoted by $\mathbb{E}^N \not\rightarrow X$.

It is not hard to see that any Ramsey set must be finite. Furthermore, it follows from compactness arguments (where we are using the Axiom of Choice) that if X is Ramsey, then in fact there must be a *finite* set S such that $S \rightarrow X$. (More about this later).

A more restricted notion is that of being r-Ramsey. This just means that a monochromatic copy of X must occur whenever the underlying set S is r-colored. In this case, we write $S \xrightarrow{r} X$. The negation of this statement is written as $S \stackrel{r}{\not\to} X$.

Conjecture 1 *For any non-equilateral triangle T, (i.e., the set of 3 vertices of T),*

$$\mathbb{E}^2 \xrightarrow{2} T.$$

Conjecture 2 *(stronger) For any partition $\mathbb{E}^2 = C_1 \cup C_2$,* every *triangle occurs (up to congruence) in C_1, or else the same holds for C_2, with the possible exception of a single equilateral triangle.*

It is easy to see how to prevent any particular equilateral triangle from occurring in a 2-coloring of the plane by using alternating color half-open strips of width equal to the altitude of the triangle.

Conjecture 3 *For any triangle T,*

$$\mathbb{E}^3 \xrightarrow{3} T.$$

There are many positive results known for triangles (e.g.,see [6]). For example, it is known that $\mathbb{E}^2 \xrightarrow{2} T$ if

(i) T has angles $(\alpha, 2\alpha, \pi - 3\alpha)$ with $0 < \alpha < \pi/3$,

(ii) T has sides (a, b, c) satisfying

$$c^2 = a^2 + 2b^2 \text{ with } a < 2b$$

On the other hand, $\mathbb{E}^3 \xrightarrow{2} T$ for any non-degenerate triangle T, and $\mathbb{E}^3 \xrightarrow{3} T$ for any right triangle T.

It is known that for any n, $\mathbb{E}^n$ can be 4-colored to prevent the degenerate $(1,1,2)$ triangle from occurring monochromatically. If Conjecture 2 is true, then this number can be reduced to 2. It is known that it cannot be reduced to 1. More generally, it is known [2] that for a and b, and any n, $\mathbb{E}^n$ can be 16-colored to prevent the degenerate $(a, b, a+b)$ triangle from occurring monochromatically.

Conjecture 4 *The number* 16 *above can be reduced.*

In other words, there is some minimum value $r < 16$ such that any degenerate $(a, b, a+b)$ triangle can be prevented from occurring monochromatically by an appropriate r-coloring of $\mathbb{E}^n$. Could this minimum value be 2?

One might well ask about the situation for even *simpler* sets than triangle. The simplest interesting set of this type is the set U consisting of *two* points a unit distance apart. The minimum number of colors needed for coloring $\mathbb{E}^n$ so that no 2 points have the same color is called the *chromatic number* of $\mathbb{E}^n$ and is denoted by $\chi(\mathbb{E}^n)$. Even for $n = 2$, the value of χ is quite mysterious. The best bounds currently known (and these haven't changed in more than 50 years!) in this case are $4 \leq \chi(\mathbb{E}^2) \leq 7$. It is certainly not necessary to point out to readers of this journal any facts concerning the history and current status of this problem (which due to E. Nelson in 1950) since the

Editor Alexander Soifer has written a scholarly treatment of the subject in this journal [11, 12, 12].

The best bounds known for $n = 3$ are $6 \leq \chi(\mathbb{E}^3) \leq 15$ due to Nechustan [19] and Bóna-Tóth [21], respectively. The best general bounds for arbitrary n are (see [29]):

$$(6/5 + o(1))^n < \mathbb{E}^n < (3 + o(1))^n$$

An interesting recent result of O'Donnell [15, 16], perhaps giving a small amount of evidence that $\chi(\mathbb{E}^2) > 4$, is the following result:

Theorem 1 *For any $g > 0$, there is a 4-chromatic unit distance graph in $\mathbb{E}^2$ with girth greater than g.*

2 Ramsey Sets

Recall that X is said to be *Ramsey* if for any number of colors r, any r-coloring of any sufficiently high-dimensional Euclidean space must always contain a monochromatic copy of X. It is not difficult to show that if X and Y are Ramsey then so is the Cartesian product $X \times Y$ is also Ramsey. This implies that every acute triangle is Ramsey. In fact, Frankl and Rödl [17] have shown that every non-degenerate simplex is Ramsey. For many years it was not known whether something as simple as the set of 5 vertices of a regular pentagon was Ramsey. This was finally settled by a striking theorem of Křiž [18]:

Theorem 2 *Suppose $X \subseteq \mathbb{E}^N$ has a transitive solvable group of isometries. Then X is Ramsey.*

In fact, the same conclusion holds under the weaker hypothesis that X has a transitive group of isometries that has a solvable subgroup with at most two orbits.

In the other direction, it was shown in [2] that any Ramsey set must lie on the surface of a sphere (in some dimension). Such sets we call *spherical.* This shows in particular that three collinear points cannot be a Ramsey set (as we have pointed out earlier). The big conjecture here is that this necessary condition is also sufficient.

Conjecture 5 *($1000). Every spherical set is Ramsey.*

Any easier conjecture is this:

Conjecture 6 *($100). Every 4-point subset of a circle is Ramsey.*

In [17], Frankl and Rŏdl define the following stronger concept: A set A is said to be *super-Ramsey* if there exist positive constants c and ϵ and subsets $X = X(N) \subset \mathbb{E}^N$ for every $N \geqslant N_0(X)$ such that $|X| < c^N$, and it is true that $|Y| < |X|/(1+\epsilon)^N$ holds for all subset $Y \subset X$ containing no copy of A

In [17] they show that every non-degenerate simplex is actually super-Ramsey. It may in fact be true that the super-Ramsey sets are just the spherical sets.

3 Sphere-Ramsey Sets

Let us denote by $S^N(\rho)$ a sphere in $\mathbb{E}^N$ with radius ρ. We will say that X is *sphere-Ramsey* if for all r, there exist $N = N(X, r)$ and $\rho = \rho(X, r)$ such that

$$S^N(\rho) \xrightarrow{r} X.$$

In this case we write $S^N(\rho) \rightarrow X$. Note that being sphere-Ramsey is a stronger condition than being Ramsey. It is easy to see that if X and Y are sphere-Ramsey then so is $X \times Y$. For a spherical set, let $\rho(X)$ denote the radius of the smallest sphere containing X. In [22], Matousěk and Rödl prove the following spherical analogue of simplices being Ramsey:

Theorem 3 *For any simplex X with $\rho(X) = 1$, any r, and any $\epsilon > 0$, there exists $N = N(X, r, \epsilon)$ such that*

$$S^N(1 + \epsilon) \xrightarrow{r} X.$$

It turns out that the "blow-up factor" of $(1 + \epsilon)$ is really needed, as the following result shows:

Theorem 4 *[9] Let $X = x_1, x_2, \ldots, x_m \subset \mathbb{E}^N$ such that:*

(i) for some nonempty $I \subseteq 1, 2, \ldots, m$, there exist nonzero $a_i, i \in I$, with

$$\sum_{i \in I} a_i x_i = 0 \in \mathbb{E}^N$$

and

(ii) for all nonempty $J \subseteq I$,

$$\sum_{j \in J} a_j \neq 0.$$

Then X is not sphere-Ramsey.

We close this section with a fundamental conjecture:

Conjecture 7 *(\$1000). If X is Ramsey then X is sphere-Ramsey.*

4 Edge-Ramsey Sets

Suppose we color the line segments in some Euclidean space and ask the natural Ramsey questions. That is, given some fixed finite configuration E of line segments in $\mathbb{E}^n$, when is it the case that for any r, there is an integer $N(E, r)$ such that for any r-coloring of the line segments in $\mathbb{E}^N$ with $N > N(E, r)$, there is always a monochromatic "copy" of E which is formed. In this case, we will say that E is *edge-Ramsey.* The basic facts which are known in this case are:

Theorem 5 *[2, 3, 4] If E is edge-Ramsey then all edges of E must have the same length.*

Theorem 6 *[9] If E is edge-Ramsey then the end-points of the edges of E must lie on two spheres.*

Theorem 7 *[9] If the end-points of E do not lie on a sphere and the graph formed by E is not bipartite then E is not edge-Ramsey.*

Theorem 8 *[24] The edge set of an n-cube is edge-Ramsey.*

Theorem 9 *[24] The edge set of a regular n-gon is not edge-Ramsey if $n \neq 6$.*

Conjecture 8 *The edge set of a regular hexagon is not edge-Ramsey.*

The situation is actually rather mysterious since Cantwell [23, 24] has pointed out that:

(i) If ABC is a $(1, 1, 2)$ triangle with $|AB| = |BC| = 1$ then the set E consisting of the two line segments AB and BC is not edge-Ramsey, even though its graph is bipartite and the three end-points A, B and C lie on two spheres.

(ii) There exist non-spherical edge sets which are edge-Ramsey.

(Big) Problem *Characterize edge-Ramsey configurations*

At present, we have no plausible conjecture.

5 Variations

In this final section we mention a number of variations of the standard Euclidean Ramsey problems.

We start with *asymmetric* Ramsey theorems. Here, we typically have two configurations, say X_1 and X_2. If it is true that for any 2-coloring of $\mathbb{E}^N$, either a copy of X_1 occurs in color 1, or a copy of X_2 occurs in color 2, then we denote this by $\mathbb{E}^N \xrightarrow{2} (X_1, X_2)$. There only sporadic results known here. We mention a few:

(a) $\mathbb{E}^2 \xrightarrow{2} (T_2, T_3)$ where T_i is any subset of $\mathbb{E}^2$ with i points, $i = 2, 3$.

(b) $\mathbb{E}^3 \xrightarrow{2} (T, Q^2)$ where T is an isoceles right triangle and Q^2 is a square.

(c) $\mathbb{E}^2 \xrightarrow{2} (P_2, T_4)$ where P_2 is a set of two points at a distance 1, and T_4 is any set of 4 points.

(d) There is a set T_8 of 8 points in $\mathbb{E}^2$ such that $\mathbb{E}^2 \stackrel{2}{\not\to} (P_2, T_8)$. (see [30]).

It is not known if there are sets with fewer than 8 points with this property.

We close with a brief discussion in which we allow *infinitely many* colors. As one might imagine, such results usually have a strong set-theoretic nature, and often depend on just which axioms one is using for set theory. For example:

Theorem 10 *[26] For all N, $\mathbb{E}^N \stackrel{\aleph_0}{\not\to} T$ where T is any fixed triangle.*

Schmerl [26] has also shown that there is a partition of $\mathbb{E}^N$ into countably many parts such that no part contains the vertices of *any* isoceles triangle.

Theorem 11 *[27] Assuming the Continuum Hypothesis, it is possible to par-*

tition $\mathbb{E}^2$ into countably many parts, none of which contains the vertices of a triangle with **rational** *area.*

Theorem 12 *[28] The existence of a partition of $\mathbb{E}^2$ into countably many sets, none of which contains the vertices of a* **right** *triangle is equivalent to the Continuum Hypothesis.*

In fact, the importance of the particular axioms being used make a surpising difference for the question of determining the chromatic number of the plane, as recently shown by Shelah and Soifer [14]. In particular, that show that if in ZFC, the Axiom of Choice is replace by some different (but equally consistent) axioms (one of which is the axiom which assets that every subset of $\mathbb{R}$ is Lebesgue measurable), then we lose the usual compactness arguments, and in fact, in this case, if every finite unit distance graph has chromatic number at most 4, then $\mathbb{E}^2$ must have chromatic strictly greater than 4.

I close with 3 classic problems, to which modest rewards have been attached.

Conjecture 9 *(Erdős-Szekeres [1]) ($1000) Any set of $2^{n-2}+1$ points in the plane in general position (i.e., having no 3 collinear points) must contain the vertices of a convex n-gon.*

If true, this value would be best possible. The best upper bound currently known is $\binom{2n-2}{n-3}+2$, due to Tóth and Valtr [31].

Conjecture 10 *(Graham [10]) ($1000) If* $A \subseteq \mathbb{N}^2$ *with* $\sum_{(x,y)\in A} 1/(x^2 + y^2) = \infty$ *then* A *must contain the* 4 *vertices of an axes-parallel square (where* $\mathbb{N}$ *denotes the set of natural numbers).*

This is a two-dimensional generalization of one of Erdős' most well-known conjectures:

Conjecture 11 *Erdős ($3000) If* $A \subseteq \mathbb{N}$ *satisfies* $\sum_{a\in A} 1/a = \infty$ *then* A *must contain arbitrarily long arithmetic progressions.*

It is not even known if this hypothesis implies that A has a 3-term arithmetic progression! Clearly, much remains to be done.

References

[1] P. Erdős and G. Szekeres, A combinatorial problem in geometry. *Compositio Math.*, **2** (1935), 463–470.

[2] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey Theory I, *J. Combinatorial Theory Ser. A* **14** (1973), 341–363.

[3] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey Theory II, *Infinite and finite sets, Colloq., Keszthely, (1973), Vol. I,* pp. 529–557. Colloq. Math. Soc. Janos Bolyai, Vol. 10, *North-Holland, Amsterdam,* 1975.

[4] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey Theory III, *Infinite and finite sets Colloq., Keszthely, (1973), Vol. I,* pp. 559–583. Colloq. Math. Soc. Janos Bolyai, Vol. 10, *North-Holland, Amsterdam,* 1975.

[5] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory.* Second edition. *John Wiley & Sons, Inc., New York,* 1990. $xii + 196$ pp.

[6] R. L. Graham, Euclidean Ramsey theory. *Handbook of discrete and computational geometry,* 153-166, CRC Press Ser. Discrete Math. Appl., *CRC, Boca Raton, FL,* 1997

[7] R. L. Graham and J. Nešetřil, Ramsey theory in the work of Paul Erd[o]s, *The mathematics of Paul Erdős, II* 193–209, Algorithms Combin.,14, *Springer, Berlin,* 1997.

[8] R. L. Graham, Recent trends in Euclidean Ramsey theory. Trends in discrete mathematics. *Discrete Math.* **136** (1994), 119–127.

[9] R. L. Graham, Euclidean Ramsey theorems on the n-sphere. *J. Graph Theory* **7** (1983), 105–114.

[10] R. L. Graham, Recent developments in Ramsey theory. *Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Warsaw, 1983),* 1555-1567, *PWN, Warsay,* 1984.

[11] Alexander Soifer, Chromatic number of the plane & its relatives. I. The problem & its history. *Geombinatorics* **12** (2003), 131–148.

[12] Alexander Soifer, Chromatic number of the plane & its relatives. II. Polychromatic number & 6-coloring. *Geombinatorics* **12** (2003), 191–216.

[13] Alexander Soifer, Chromatic number of the plane & its relatives. III. Its future. *Geombinatorics* **13** (2003), 41–46

[14] S. Shelah and A. Soifer, Axiom of choice and chromatic number of the plane. *J. Combin. Theory Ser. A* **103** (2003), 387–391.

[15] P. O'Donnell, Arbitrary girth, 4-chromatic unit distnace graphs in the plane. I. Graph description. *Geombinatorics* **9** 145–152.

[16] P. O'Donnell, Arbitrary girth, 4-chromatic unit distnace graphs in the plane. II. Graph embedding. *Geombinatorics* **9** 180–193.

[17] P. Frankl and V. Rödl, A partition property of simplices in Euclidean space. *J. Amer. Math. Soc.,* **3**, (1990), 1–7.

[18] I. Kříž, Permutation groups in Euclidean Ramsey theory. *Proc. Amer. Math. Soc.* **112** (1991), 899–907.

[19] O. Nechushtan, On the space chromatic number. *Discrete Math.* **256** (2002), 499–507.

[20] P. Frankl and V. Rödl, Intersection theorems with geometric consequences. *Combinatorica,* **1** (1931), 357–368.

[21] R. Radoičič and G. Tŏth, A note on the chromatic number of $\mathbb{R}^3$ (to appear).

[22] J. Matoušek and V. Rödl, On Ramsey sets on spheres. *J. Combin. Theory Ser. A,* **1995**, 30–44.

[23] K. Cantwell, Finite Euclidean Ramsey theory. *J. Combin. Theory Ser. A,* **73** (1996), 273–285.

[24] K. Cantwell, Edge-Ramsey theory. *Discrete Comput. Geom.,* **15** (1996), 341–352.

[25] P. Schmitt. Problems in discrete and combinatorial geometry. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry,* vol. A. *North-Holland, Amsterdam,*, 1993.

[26] J. H. Schmerl. Triangle-free partitions of Euclidean space. *Bull. London Math. Soc,* **27** (1994), 483–486.

[27] K. Kunen (personal communication).

[28] P. Erdős and P. Komjăth. Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Discrete Comput. Geom.,* **5** (1990), 325–331.

[29] H. T. Croft, K. J. Falconer and R. K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, New York, 1991.

[30] G. Csizmadia and G. Tóth, Note on a Ramsey-type problem in geometry. *J. Combin. Theory Ser. A,* **65** (1994), 302–306.

[31] G. Tóth and P. Valtr. Note on the Erdős-Szekeres theorem. In J. Pach, editor, Erdős Memorial Issue, *Discrete Comput. Geom.,* **23** (1998), 457–459.